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Efficiency with Natural Resources

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Keywords： OLG, dynamic equilibrium, efficiency, natural resources

JEL codes： O13; O40; Q20; Q30

Efficiency with Natural Resources*

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Abstract

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1 Introduction

In resource economics, the overlapping generations (OLG) model is a fundamental tool for analyzing interactions between different generations in resource utilization. It is well known that in OLG economies, even without natural resources, equilibrium can be Pareto-inefficient. This means that the market mechanism does not always result in an efficient allocation of resources. When natural resources are introduced, the problem becomes even more complex.

A fundamental question arises: Given an equilibrium in an OLG economy, is it Pareto-efficient? How can we assess the Pareto efficiency of such an equilibrium? Are there appropriate tools or criteria for doing so? This issue is not only of academic importance but also crucial for policymakers to determine whether market intervention is necessary and, if so, what form that intervention should take. Only when the efficiency issue is identified can we evaluate the role of markets in resource allocation and consider the need for government intervention.

Many researchers have explored this topic. Notably, Malinvaud (1953), Cass (1972), and Mitra (1978) introduced well-known criteria for dynamic efficiency.[2] Balasko and Shell (1981) analyzed Pareto efficiency in pure exchange OLG economies through price mechanisms, Homburg (1992) proposed an income-based criterion, and Abel et al. (1989), referred to as AMSZ (1989) in the sequel, developed a net dividend criterion for Pareto efficiency. However, most of these studies exclude natural resources, with the exception of Mitra (1978), who only considered exhaustible resources. For a broader class of natural resources, no widely applicable criteria exist for determining the Pareto efficiency

[2] Related to but distinct from Pareto efficiency.

of equilibrium in OLG economies. We aim to fill this gap by establishing such criteria for more general natural resource scenarios.

In resource economics, most previous researchers on Pareto efficiency in OLG economies have focused on models with natural resources but without physical capital, primarily analyzing steady-state equilibrium, which is far simpler than dynamic equilibrium. However, when both natural resources and physical capital are considered simultaneously, the problem becomes significantly more complex. In reality, physical capital and natural resources typically coexist in production processes, and their interrelationship has a substantial impact on output. Therefore, it is essential to include both physical capital and natural resources in the analysis. Furthermore, focusing solely on steady-state equilibrium limits the analysis to long-run outcomes, whereas dynamic equilibrium captures the entire process of economic development. Dynamic equilibrium analysis is especially important when a steady state does not exist, but a dynamic equilibrium does.

In this paper, we consider OLG economies with natural resources and physical capital, aiming to establish criteria for assessing the efficiency of equilibrium. Our core approach is to compare the growth rates of capital, income, and total asset value with the interest rate to determine whether the economy is efficient. Intuitively, if the capital growth rate exceeds the interest rate, it may lead to overaccumulation of capital, resulting in inefficiency. Similarly, if the growth rates of income or total asset value surpass the interest rate, this suggests that income or wealth has not been fully utilized, leading to waste and inefficiency. Conversely, if these growth rates are lower than the interest rate, overaccumulation or waste will not occur, indicating efficiency. Of course, for greater accuracy, certain detailed preconditions must be considered.

The structure of this paper is as follows: First, we conduct a systematic review of the relevant literature. Second, we construct the model framework and present several general criteria. Third, we apply these criteria to three specific examples. Each of these examples represents an interesting model, illustrating some important features of resource utilization. Agnani et al. (2005) introduced a simplified version of the first model, focusing solely on exhaustible resources, whereas the second and third models are original to this paper.

More concretely, in each example, the utility function is log-linear, whereas the resource regeneration function and production function vary. In the first example, the resource regeneration function is linear, and the production function follows a Cobb–Douglas form. In the second example, the resource regeneration function remains linear, but the production function is constant elasticity of substitution (CES)-type beyond Cobb–Douglas. In the third example, the resource regeneration function is logistic, whereas the production function is of Cobb-Douglas. Along the way, we also discuss issues of sustainability and equilibrium stability.

Additionally, through these examples, we address the question: Does the inclusion of natural resources improve economic efficiency compared to an economy without natural resources? We find that the relationship between natural resources and capital plays a crucial role. Generally, when natural resources and capital are complementary, the inclusion of natural resources enhances economic efficiency, potentially transforming previously inefficient economies into efficient ones. However, when natural resources and capital are substitutable, their interaction in production is less tightly coordinated, and the inclusion of natural resources may fail to improve economic inefficiency. In such cases, slower extraction of natural resources tends to promote economic efficiency.

It is important to note that this paper primarily focuses on determining whether a given equilibrium is Pareto-efficient, rather than exploring the existence of equilibrium under general conditions. While the existence of equilibrium is certainly a significant issue, typically, existence and efficiency can be analyzed separately. For the three examples discussed, of course, we first need to identify the equilibrium and then evaluate its efficiency.

Developing general criteria for assessing the efficiency of equilibrium in OLG economies is already a challenging task. Due to space limitations, we do not address government intervention in cases of inefficiency. Additionally, we only consider deterministic scenarios and do not tackle the complexities arising from uncertainty.

2 Literature Review

In this section, we provide a brief literature review on the topic.

2.1 Dynamic Efficiency in Ramsey Economy

To date, the most significant works, considered as classical, are as follows.

Malinvaud (1953) provides a criterion later known as the Malinvaud condition for a program to be dynamically efficient: the present value of capital tends to zero. It is a kind of transversality condition.

Cass (1972) proposes a result later known as the Cass criterion for a program to be dynamically efficient: the sum of the reciprocals of the discount factors is divergent.

Benveniste and Gale (1975), under some conditions on the production function, extend the Cass criterion: a program is dynamically efficient, if the sum of the reciprocals of the norms of present value of capital is divergent.

Mitra (1978), considering exhaustible resources, under the assumption that the resource is important in production, proposes a necessary and sufficient condition for a program to be dynamically efficient: it is competitive and satisfies a Malinvaud-type condition: the present value of total assets tends to zero.

2.2 Pareto Efficiency in OLG Economy without Natural Resources

The following works are the most influential and serve as benchmarks in this field.

Balasko and Shell (1980, 1981a, 1981b), inspired by Cass's work, prove that in a pure exchange OLG economy, under certain assumptions, an equilibrium is Pareto-inefficient if and only if the sum of the reciprocals of the price norms converges.

Wilson (1981), for a pure exchange economy of infinite time horizon which combines infinite-lived agents with finite-lived agents, under certain additional assumptions, proposes a sufficient condition for the Pareto efficiency of equilibrium: the sum of values of all initial endowments converges.

AMSZ (1989) introduces a net dividend criterion in a stochastic environment.

Geanakoplos et al. (1991), focusing on the pure exchange OLG economy, set a sufficient condition for Pareto efficiency of equilibrium: the first generation must possess a resource that consistently contributes to income across periods.

Homburg (1992) proposes a sufficient condition for Pareto efficiency of equilibrium: the present value of wages vanishes. This condition is further clarified in Croix et al. (2004).

Tirole (1985), in an OLG setting with both productive and nonproductive assets, demonstrates the existence of multiple equilibria with varying levels of efficiency and bubbles. In assessing the efficiency of an equilibrium, he compares the population growth rate with the interest rate.

2.3 Pareto Efficiency in OLG Economy with Natural Resources

In resource economics, regarding Pareto efficiency in OLG economies, the most significant works are as follows.

Kemp and Long (1979) provide an OLG model with an exhaustible resource but without capital, in which the production function is not homogeneous of degree one, and the resource is not essential for production. The unique equilibrium is inefficient, in which nothing is extracted.

Rhee (1991) examines an OLG economy with land and provides a sufficient condition for the Pareto efficiency of equilibrium: land is important in production, meaning that the income share of land does not diminish. He also demonstrates that this condition is not necessary by offering a counterexample.

Olson et al. (1997) explore an OLG economy with an exhaustible resource but without physical capital. They prove that the resource is ultimately depleted, and that the equilibrium is Pareto-efficient without assuming that the resource is important in production.

Krautkraemer (1999), like Olson et al. (1997), examines an OLG economy with a natural resource but without physical capital where the utility function is additive log. He particularly emphasizes on the case where the regeneration function of the resource is logistic. He highlights that when the resource's output share is relatively small, overaccumulation of the resource may occur, leading to a Pareto-inefficient steady-state equilibrium.

Koskela et al. (2002) incorporate a renewable resource into an OLG model without capital and with quasi-linear preferences. Under certain conditions, they demonstrate the existence of two steady-state equilibria: one stable and the other unstable. The unstable one is always Pareto efficient, while the stable one may or may not be Pareto-efficient.

Agnani et al. (2005) consider an OLG economy with an exhaustible resource and physical capital, where production function is of Cobb–Douglas, and the utility function is log-linear. Under the assumption that the economy follows a balanced growth path, they show that the equilibrium is socially optimal and thus Pareto-efficient, and the economy will contract if the labor share is relatively small.

Farmer et al. (2017) examine an OLG economy with a renewable resource but without capital, where the utility function is log-linear, the production function is Cobb–Douglas, and the resource regeneration function is logistic. They incorporate varying harvest costs and show that, with inversely stock-dependent costs and certain assumptions on model parameters, a unique steady-state equilibrium exists, which is asymptotically stable. Somewhat surprisingly, the steady-state equilibrium can be Pareto-efficient even when the own rate of return on the resource stock is negative.

We observe that there is no general criterion for assessing the Pareto efficiency of equilibria.

Finally, we would like to point out that many researchers, including Hellwig (2024), have discussed the issue of efficiency in stochastic OLG models. However, their primary focus is on addressing uncertainty, and their results do not cover all of the findings mentioned in the literature review above for the deterministic scenario. Our aim in this paper is to establish broadly applicable criteria for the deterministic case that can encompass the results of previous works.

3 Model Setup

We begin with some preliminary notations. Let $\mathbb{R}_+$ be the set of positive numbers, $\mathbb{N}$ ($\mathbb{N}_+$) the set of nonnegative (positive) integers, and $\mathbb{N}_- = \{-1\} \cup \mathbb{N}$.

For any two positive dynamic variables x_t and y_t, we use $x_t \sim y_t$ to indicate that y_t/x_t converges to some positive number, as $t \to \infty$.

3.1 The Economy

Consider a two-period OLG economy with natural resources, existing at all points in time within $\mathbb{N}$.

Population: At any time $t \in \mathbb{N}$, a new generation (generation-t) of population N_t is born, living for two periods. Each individual of generation-t has utility function $U(a_t, b_{t+1})$, where a_t and b_{t+1} represent their consumption at t and $t+1$ respectively, and U is smooth, concave, and strictly increasing with respect to every element.

Furthermore, at $t = 0$, there is an original generation of size N_{-1}. Each member of this generation lives for just one period with utility function $u(b_0)$, where u is strictly increasing, smooth, and concave, and b_0 is their consumption at $t = 0$.

Endowments: Every young individual is endowed with one unit of labor. Members of the original generation evenly share the physical capital $\overline{K}_0 > 0$ and natural resource $\overline{S}_0 > 0$ (be it renewable or nonrenewable).

Firms: At each time $t \in \mathbb{N}$, there is only one sector comprising numerous homogeneous firms sharing an identical technology represented by production function

$$Y = F^t(K, L, R),$$

where Y is the output of the final good, and K, L and R are the inputs of factors: the physical capital, labor, natural resources, respectively, F^t is the production function at time t, which is first-order homogeneous, smooth, concave, and strictly increasing with respect to each element. The production functions may change due to technological progress. The final good can either be consumed or invested in physical capital. For simplicity, the depreciation rate for the physical capital is assumed to be 1.

Natural resources: Each natural resource, viewed as a unified entity, is extracted and sold. They are not physically divided among owners. Instead, owners have shared property rights and therefore equally split the revenue derived from these resources[3]. Harvesting these resources is cost-free.

Regarding resource transaction and dynamics, we make assumptions as follows[4]:

At the start of each period t, the natural resource (with a stock of S_t) is held by older adults with even property rights. A portion of the resource, R_t, is extracted and sold to the firms, and the remaining resources $S_t - R_t$ are sold to young people with even property rights. By the start of the next period, $t+1$, the resource stock grows to $G(S_t - R_t)$. The function G describes resource regeneration, being smooth, concave, and nonnegative, defined over $[0, \infty)$, with properties $G(0) = 0$, $G'(0) \in (0, \infty]$, $G'(x) > 0$, $\forall x > 0$.

In particular, $G(x) = x$ refers to the case of exhaustible (nonrenewable) resources.

The dynamics of the resource are described by

$$S_{t+1} = G(S_t - R_t).$$

Assume that all markets are completely competitive, and every young individual has perfect foresight regarding the price system in the next period.

[3]Tirole(1985) and Rhee(1991), among others, use such a treatment.
[4]Farmer(2000), among others, use an alternative approach.

3.2 Efficiency and Social Optimality

The main concern in this paper is Pareto efficiency. Other two related concepts are dynamic efficiency and social optimality. We discuss them separately.

We first give the concepts of allocation and program.

Allocation: A sequence of nonnegative vectors $\{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$ is called an allocation, if it satisfies the conditions of feasibility: $K_0 = \overline{K}_0$, $S_0 = \overline{S}_0$, and for any $t \in \mathbb{N}$,

$$N_t a_t + N_{t-1} b_t + K_{t+1} \le F^t(K_t, N_t, R_t),$$

$$S_{t+1} = G(S_t - R_t).$$

Denote the set of all allocations by $\mathscr{A}$. For any allocation $\mathbb{A} = \{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, and any $t \in \mathbb{N}_-$, denote the utility of generation-t under $\mathbb{A}$ as $U_t(\mathbb{A})$.

Program: A sequence of nonnegative vectors $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$ is called a program, if it satisfies the conditions of feasibility: $K_0 = \overline{K}_0$, $S_0 = \overline{S}_0$, and for any $t \in \mathbb{N}$,

$$C_t + K_{t+1} \le F^t(K_t, N_t, R_t),$$

$$S_{t+1} = G(S_t - R_t).$$

Denote the set of all programs by $\mathscr{P}$.

For any allocation $\{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, the program $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, where

$$C_t = N_t a_t + N_{t-1} b_t, \quad \forall t \in \mathbb{N},$$

is called its corresponding program. Here, a_t, b_t, and C_t are the consumption of each young man, the consumption of each old man, and the aggregate consumption at time t, respectively.

Pareto improvement: An allocation $\mathbb{A}$ is Pareto-improved by another allocation $\mathbb{A}'$ if

$$U_t(\mathbb{A}) \le U_t(\mathbb{A}'), \quad \forall t \in \mathbb{N}_-,$$

with at least one inequality being strict.

Pareto efficiency: An allocation is Pareto-efficient, if it cannot be Pareto-improved by any allocation.

The other concept is dynamic efficiency.

Dynamic improvement: A program $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$ is dynamically improved by another program $\{C'_t, K'_t, S'_t, R'_t\}_{t\in\mathbb{N}}$, if

$$C_t \leq C'_t, \quad \forall t \in \mathbb{N},$$

with at least one inequality being strict.

Dynamic efficiency[5]: A program is dynamically efficient if it cannot be dynamically improved by any program.

An allocation is dynamically efficient if its corresponding program is dynamically efficient.

Clearly, dynamic efficiency is weaker than Pareto efficiency. But, the converse is not universally true, even if the allocation is an equilibrium allocation (regarding equilibrium, see subsection 3.3 below). Here is a counterexample, the essence of which is the same as in the Hilbert's infinite Hotel paradox.

There is no natural resource, no population growth, and no technological progress. The production function is $F(K, L) = K + L$, the utility function is $U(a, b) = a + b$, and the initial endowment of capital of the ancestor is $K_0 = 1$. Then the equilibrium allocation $(a_t, b_t, k_t)_{t\in\mathbb{N}}$ (the corresponding price system is $r_t \equiv 0$, $\omega_t \equiv 1$) is dynamically efficient, where

$$a_t = 1, \quad \forall t \in \mathbb{N},$$

$$b_0 = 1, \quad b_t = 0, \quad \forall 1 \leq t \in \mathbb{N},$$

$$k_0 = 1, \quad k_t = 0, \quad \forall 1 \leq t \in \mathbb{N}.$$

But it is not Pareto-efficient, because it can be Preto-improved by the following allocation $(a'_t, b'_t, k'_t)_{t\in\mathbb{N}}$:

$$a'_t = 0, \quad \forall t \in \mathbb{N},$$

$$b'_0 = 2, \quad b'_t = 1, \quad \forall 1 \leq t \in \mathbb{N},$$

$$k'_0 = 1, \quad k'_t = 0, \quad \forall 1 \leq t \in \mathbb{N},$$

[5] Some authors call it the dynamic efficiency in the aggregate. See, for example, Miao (2020).

which is itself Pareto-efficient[6].

The third concept is social optimality, which is concerned with measuring social welfare. The social welfare is typically measured by a social welfare functional. In most cases, this functional takes the form of a weighted sum of utilities across generations[7]: given a sequence of positive numbers as weights $\lambda = (\lambda_t)_{t\in\mathbb{N}_-}$, for any $\mathbb{A} \in \mathscr{A}$, the social welfare functional is defined as

$$W_\lambda(\mathbb{A}) = \sum_{t=-1}^{\infty} \lambda_t U_t(\mathbb{A}).$$

The most commonly used weights follow an exponential form: for any $t \in \mathbb{N}_-$, $\lambda_t = \varepsilon^t$, where $\varepsilon \in (0,1)$ is known as the social discount factor.

In this paper, we only consider the specific weights of this exponential form, and simply denote it as W_ε.

In particular, if for any $t \in \mathbb{N}$, the utility function for generation-t is of the form $U(a_t, b_{t+1}) = u(a_t) + \rho u(b_{t+1})$, and the utility function for the ancestor is $\rho u(b_0)$, where u is some smooth, concave, and strictly increasing function, then, for any $\varepsilon \in (0,1)$, the social welfare functional W_ε can be simplified to a reduced form: for any $\mathbb{A} = (a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}} \in \mathscr{A}$,

$$W_\varepsilon(\mathbb{A}) = \rho u(b_0) + \sum_{t=0}^{\infty} \varepsilon^{t+1} \left(u(a_t) + \rho u(b_{t+1})\right) = \sum_{t=0}^{\infty} \varepsilon^t (\varepsilon u(a_t) + \rho u(b_t)).$$

Social optimality: An allocation is socially optimal with respect to a social welfare functional W_ε for some $\varepsilon \in (0,1)$ if it maximizes W_ε over $\mathscr{A}$.

Obviously, Pareto efficiency is weaker than social optimality. But the converse is not universally true, even under arbitrary weights.

Social optimality can serve as a useful tool for assessing Pareto efficiency, while also holding intrinsic significance in its own right. From a societal per-

[6]This assertion can be proved by a lemma similar to Lemma 3 in the Appendix. It states that an allocation can be Pareto-improved if and only if the ancestor can be made strictly better off without making anyone else worse off. In allocation $(a'_t, b'_t, k'_t)_{t\in\mathbb{N}}$, the ancestor's consumption is 2, which is already the maximum of the output at time $t = 0$ and of course cannot be improved any more.

[7]In certain scenarios, such a social welfare functional may not be well defined. In these instances, an alternative approach, such as the *overtaking criterion*, may be employed instead.

spective, social optimality offers a framework for determining whether a given allocation is desirable. This criterion operates at a higher level than the Pareto principle, which is often regarded as the most fundamental form of optimality. Consequently, many researchers who address Pareto efficiency also consider social optimality where appropriate. In this paper, we apply the concept of social optimality in some specific examples. However, in cases where it is difficult to construct a suitable social welfare functional, we refrain from further analysis.

3.3 Equilibrium

With the final good as the numéraire (with price set to 1), the prices of the physical capital, labor and the natural resource at time $t \in \mathbb{N}$ are denoted as r_t, ω_t, and p_t, respectively, and the consumption of each young individual and the consumption of each old individual at time t are denoted as a_t and b_t, respectively.

Equilibrium: A price system and an allocation, $\{r_t, \omega_t, p_t; a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, with $(1+r_t, \omega_t, p_t, a_t, b_t, K_t, S_t, R_t) \in \mathbb{R}^8_+$ for any $t \in \mathbb{N}$, is called a dynamic equilibrium (or simply an equilibrium), if for any $t \in \mathbb{N}$,

$$\begin{aligned} & (a_t, b_{t+1}, K_{t+1}/N_t, (S_t - R_t)) \\ \in \quad & \arg\max_{(a,b,s,X)} \{U(a,b) | a + s + p_t X/N_t \leq \omega_t; b = (1 + r_{t+1})s + p_{t+1}G(X)/N_t\}; \end{aligned}$$

$$(K_t, N_t, R_t) \in \arg\max_{(K,L,R)} F^t(K, L, R) - (1 + r_t)K - \omega_t L - p_t R;$$

$$K_{t+1} = F^t(K_t, N_t, R_t) - N_t a_t - N_{t-1} b_t, \quad S_{t+1} = G(S_t - R_t).$$

It's easy to verify that along the equilibrium path $\{r_t, \omega_t, p_t; a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, it holds that for any $t \in \mathbb{N}$,

$$1 + r_{t+1} = \frac{p_{t+1}G'(S_t - R_t)}{p_t}, \tag{1}$$

which is the no-arbitrage condition, implying that the rates of return on investments in any assets (including physical capital and natural resource) are equal.

This no-arbitrage condition can be referred to as the generalized Hotelling rule. It reduces to the classical Hotelling rule (Hotelling(1931)) when $G(x) = x$, corresponding to the case of exhaustible resources.

Existence of equilibrium: Concerning the existence of the equilibrium, it is easy to see that the following basic assertion holds true, which we present as a lemma.

Lemma 1. An equilibrium exists if and only if the following system of equations for $(a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ with $K_0 = \overline{K}_0, S_0 = \overline{S}_0$ has a positive solution:

$$\begin{aligned}
&F_K^t(K_{t+1}, N_{t+1}, R_{t+1}) = \frac{F_R^{t+1}(K_{t+1}, N_{t+1}, R_{t+1})}{F_R^t(K_t, N_t, R_t)} G'(S_t - R_t),\\
&F_K^t(K_{t+1}, N_{t+1}, R_{t+1}) = \frac{U_a(a_t, b_{t+1})}{U_b(a_t, b_{t+1})},\\
&F^t(K_t, N_t, R_t) = N_t a_t + N_{t-1} b_t + K_{t+1},\\
&N_{t-1} b_t = K_t F_K^t(K_t, N_t, R_t) + S_t F_R^t(K_t, N_t, R_t),\\
&S_{t+1} = G(S_t - R_t).
\end{aligned}$$

The first equation is the previously mentioned generalized Hotelling rule (1); the second reflects the equality of MRS (marginal rate of substitution) and MRT (marginal rate of transformation) in each period; the third is the feasibility condition; the fourth indicates that older people consume all of their assets; and the fifth describes the dynamic equation for the resource stock.

Because $(N_t)_{t\in\mathbb{N}}$ is given exogenously, and $(a_t, b_t)_{t\in\mathbb{N}}$ can be derived from $(K_t, S_t, R_t)_{t\in\mathbb{N}}$, then, the above system of equations for $(a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ can be equivalently transformed to a system of equations for $(K_t, S_t, R_t)_{t\in\mathbb{N}}$. According to the implicit function theorem, under certain conditions (e.g., the corresponding Jacobian matrix is nondegenerate), explicit recursive equations can be derived:

$$\begin{aligned}
K_{t+1} &= \varphi(K_t, S_t, R_t, N_t, N_{t+1}),\\
S_{t+1} &= G(S_t - R_t),\\
R_{t+1} &= \psi(K_t, S_t, R_t, N_t, N_{t+1}),
\end{aligned}$$

where φ, ψ are some functions. In this case, an equilibrium exists, if and only if there exists an $R_0 > 0$, ensuring the entire trajectory of $(K_t, S_t, R_t)_{t\in\mathbb{N}}$ remains in $\mathbb{R}^3_+$[8].

Although this lemma is quite evident, it points to a fundamental approach for identifying equilibrium. In the subsequent three examples in section 5, the search for equilibrium follows this approach.

3.4 Categories of OLG Economies

In this subsection, we emphasize that, under the general framework outlined above, all economies can be categorized into two groups: Group I: economies where no equilibrium exists; and Group II: economies where equilibria exist (which may be unique).

Both of these groups are non-empty. To illustrate, consider the following two OLG economies, which differ only in their production functions. The initial capital and resource stocks are given, with no technological progress or population change. The utility function is $U(a, b) = \ln a + \frac{1}{2}\ln b$, and the resource regeneration function is $G(x) = 3x$. The production functions for each economy are as follows:

Economy 1: $F(K, L, R) = (KLR)^{1/3}$;

Economy 2: $F(K, L, R) = 3(K^{-1} + L^{-1} + R^{-1})^{-1}$.

Then Economy 1 has a unique equilibrium, whereas Economy 2 has no equilibrium[9].

Next, let us consider an economy belonging to Group II, which means that equilibria exist in this economy. Now, if we take any one of these equilibria, we ask: Is it efficient?

To reframe the issue: given an equilibrium, is it efficient? How can we assess its efficiency? Is there a criterion for evaluating it? These are the primary concerns of this paper.

[8] At times, this dynamical system can be further transformed to a dynamical system of $(k_t, s_t, z_t)_{t\in\mathbb{N}}$, where k_t, s_t, z_t are the capital, resource stock, and resource extraction per effective labor.

[9] See Proposition 8.

From this perspective, assuming the existence of equilibrium in a general OLG economy is reasonable. However, for a specific and concrete economy, whether it belongs to Group I or Group II is already determined, so assuming the existence of equilibrium is no longer appropriate.

We do not conduct further discussion about the existence of the equilibrium in general framework[10].

For simplicity, we say the equilibrium is dynamically efficient (Pareto-efficient), if the corresponding equilibrium allocation is dynamically efficient (Pareto-efficient).

4 General Results

Given an equilibrium of the above economy $\{r_t, \omega_t, p_t; a_t, b_t; K_t, S_t, R_t\}_{t\in\mathbb{N}}$, the question we are concerned with is: is it efficient? In order to answer this question, we first need to introduce some basic notations. Afterward, we will provide some criteria to assess efficiency.

For any $t \in \mathbb{N}$, denote the total output as

$$Y_t = F^t(K_t, N_t, R_t).$$

For any $t \in \mathbb{N}_+$, define the market discount factor from time t to time 0 as

$$D_t = \prod_{s=1}^{t}(1 + r_s)^{-1},$$

and set $D_0 := 1$. For convenience, let $D_{-1} = 1 + r_0$.

We know that by the generalized Hoteling rule, for any $t \in \mathbb{N}$,

$$D_{t+1}p_{t+1}G'(S_t - R_t) = D_t p_t.$$

For any $t \in \mathbb{N}$, denote the total value of assets held by all the old people at time t as

$$V_t = (1 + r_t)K_t + p_t S_t,$$

[10] For further studies concerning the existence of equilibrium, refer to Galor (1989), Geanakoplos (2008), among others.

the combined total investments made by all the young people of generation-t as

$$M_t = K_{t+1} + p_t(S_t - R_t),$$

the total income from both labor and investment in natural resources of generation-t as

$$I_t = \omega_t N_t + \frac{p_{t+1}}{1+r_{t+1}} S_{t+1} - p_t(S_t - R_t).$$

At time t, the old people hold the assets (K_t, S_t). Through market transactions, they can obtain the total revenue V_t and consume it. Clearly, $V_t = N_{t-1}b_t$.

For any $t \in \mathbb{N}$, when considering society as a whole, define the dividend as

$$Z_t = V_t - M_t = (1+r_t)K_t - K_{t+1} + p_t R_t.$$

For any $t \in \mathbb{N}$, denote the growth rate of total income and the growth rate of the physical capital stock (the growth rate of capital, for short) at time t by

$$i_t = \frac{I_t}{I_{t-1}} - 1, \quad j_t = \frac{K_{t+1}}{K_t} - 1,$$

respectively.

Of course, if there is no natural resource, the above concepts reduce respectively to

$$V_t = (1+r_t)K_t, \quad M_t = K_{t+1}, \quad I_t = \omega_t N_t, \quad Z_t = (1+r_t)K_t - K_{t+1}.$$

4.1 Main Criteria

We discuss the dynamic efficiency and Pareto efficiency separately.

4.1.1 Dynamic Efficiency

We provide a criterion for dynamic efficiency.

Theorem 1. The equilibrium is dynamically efficient, if

$$\varliminf_{t\to\infty} D_t V_t = 0. \tag{2}$$

Remark 1. Condition (2) is the Malinvaud-type condition. Malinvaud (1953) originally formulates it for cases without natural resources. Mitra (1978) extends Malinvaud's result to cases with exhaustible resources. We further extend it to cases with any type of natural resources. This condition means that the total wealth is eventually exhausted.

In order to get the converse of the above theorem, we make the following assumptions:

A1. The regeneration function for the natural resource is linear.

A2. The natural resource is important in production relative to labor, meaning that[11]

$$\varliminf_{t\to\infty} \frac{p_t R_t}{\omega_t N_t} > 0.$$

Theorem 2. Under assumptions **A1,A2**, if the equilibrium is dynamically efficient, then,

$$\varliminf_{t\to\infty} D_t V_t = 0.$$

Remark 2. Mitra (1978) obtains this result for the case of exhaustible resources. We extend it to cases involving natural resources with arbitrary linear regeneration functions.

4.1.2 Pareto Efficiency

We provide a criterion for Pareto efficiency.

Theorem 3. The equilibrium is Pareto-efficient if

$$\varliminf_{t\to\infty} D_t \omega_t N_t = 0. \tag{3}$$

Remark 3. Homburg (1992) first introduced condition (3) without considering natural resources[12]. We extend it to cases involving natural resources. The condition means that all income derived from labor is eventually exhausted.

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[11] A stronger version of this condition is

$$\inf_{t\in\mathbb{N},(K_t,N_t,R_t)\in\mathbb{R}^3_+} \frac{R_t F^t_R(K_t,N_t,R_t)}{F^t(K_t,N_t,R_t)} > 0.$$

That is, the minimum income share of natural resource at all times is away from zero.

[12] See Theorem 1 on p.9 in Homburg (1992), which excludes natural resources, although he later addresses land in subsequent chapters.

To some extent, we can say that the criterion in Theorem 3 involves comparing the growth rate of income from labor (e.g., wages) with the interest rate. Clearly, condition (3) is weaker than

$$\overline{\lim_{t\to\infty}} \frac{1+i'_t}{1+r_t} < 1,$$

where $i'_t = (\omega_t N_t)/(\omega_{t-1}N_{t-1}) - 1$ is the growth rate of total wages.

And obviously, condition (3) is also weaker than the Wilson-type condition:

$$\sum_{t=0}^{\infty} D_t Y_t < \infty.$$

We now provide a criterion for Pareto inefficiency. We need the following assumption.

A3. The technological progress is Harrod-neutral, that is, for any $t \in \mathbb{N}$, $F^t(K, L, R) = F(K, B_t L, R)$ for some function F, which is first-order homogeneous, smooth, concave, and strictly increasing with respect to each element; and $B_t = (1+\nu)^t$, $N_t = (1+n)^t$, where $\nu \geq 0$, $n > -1$ are constants; $\sup_t z_t < \infty$; $\lim_{k\to\infty} f_k(k, z) < \mu =: (1+\nu)(1+n)$ uniformly for z in any bounded interval; and either $\liminf_{t\to\infty} k_t > 0$, or $f_k(0, z) < \infty$ for any $z \geq 0$, where $f(k, z) = F(k, 1, z)$, $k_t = K_t/(B_t N_t)$, $z_t = R_t/(B_t N_t)$[13].

Theorem 4. Under assumptions **A3**, the equilibrium is Pareto-inefficient if

$$\underline{\lim_{t\to\infty}} \frac{1+j_t}{1+r_t} > 1. \tag{4}$$

Remark 4. This result implies roughly that if the growth rate of the capital stock exceeds the interest rate in the long run, the economy experiences overaccumulation of capital, leading to inefficiency. This idea first appeared in Phelps (1961) in the context of a Ramsey economy without natural resources. For the OLG economy, AMSZ (1989) initially presented this criterion, also without considering natural resources. We extend it to cases where natural resources are taken into account under some conditions on the production functions.

[13]The condition $\sup_t z_t < \infty$ means that the resource extraction per effective labor is bounded above. A stronger version is $\sup_t S_t/(B_t N_t) < \infty$, which is determined exogenously by the regeneration function, technological progress, and the population growth.

Remark 5. (Further discussion on AMSZ (1989)) AMSZ (1989) introduces the net dividend criterion for Pareto inefficiency, stating that an equilibrium allocation is Pareto-inefficient if

$$\frac{Z_t}{M_t} \le -\epsilon, \quad \forall t \in \mathbb{N}, \tag{5}$$

for some $\epsilon > 0$. This is equivalent to

$$\frac{1+j_t}{1+r_t} \ge 1+\epsilon', \quad \forall t \in \mathbb{N},$$

for some $\epsilon' > 0$.

Clearly, our condition (4) is weaker than (5) of AMSZ (1989). Instead of requiring the capital growth rate to consistently exceed the interest rate throughout the entire process of economic development, our condition only demands this in the long run, as time approaches infinity. What is the intuition behind our result? From a certain point in time, say T, onward, if the capital growth rate exceeds the interest rate, this is sufficient to ensure inefficiency. There is no need to impose this condition from the very beginning (i.e., $T = 0$). The portion of the equilibrium allocation before T can remain unchanged, while the allocation after T can be Pareto improved by reducing capital and increasing consumption. This improvement is possible as long as the capital growth rate continues to exceed the interest rate from time T onward.

AMSZ (1989) also presents the net dividend criterion for Pareto efficiency, stating that an equilibrium allocation is Pareto-efficient if

$$\frac{Z_t}{M_t} \ge \epsilon, \quad \forall t \in \mathbb{N}, \tag{6}$$

for some $\epsilon > 0$. This is equivalent to

$$\frac{1+j_t}{1+r_t} \le 1-\epsilon', \quad \forall t \in \mathbb{N},$$

for some $\epsilon' > 0$. However, Chattopadhyay (2008) disproves this criterion by providing a counterexample[14].

[14] After adding an additional condition, condition (6) is really sufficient for the Pareto efficiency of the equilibrium. See Corollary 3 below.

At the end of this subsection, we attempt to investigate the converse of Theorem 3 to some extent.

Theorem 3 gives a criterion of Pareto efficiency in terms of income from labor (i.e., the wages), but, it is not necessary[15].

Now, we try to provide a criterion for Pareto inefficiency in terms of income. However, we find that we need to modify the income from labor to the total income, and compare the growth rate of the total income with the interest rate, which is similar to the criterion in Theorem 4 comparing the growth rate of capital and the interest rate.

Additionally, to derive a concise form of such a criterion, here we only consider the case of log-linear utility function[16]. But the regeneration and production functions remain arbitrary, ensuring that the result is still relatively general.

Theorem 5. Suppose $U(a, b) = \ln a + \rho \ln b$, where $\rho \in (0, 1)$ is a constant. Then, the equilibrium is Pareto-inefficient if

$$\varliminf_{t\to\infty} \frac{1+i_t}{1+r_t} > 1. \tag{7}$$

Remark 6. Condition (7) implies that if total income grows too rapidly, eventually surpassing the interest rate in the long run, valuable resources will remain underutilized, leading to inefficiency. This introduces a new criterion for assessing Pareto inefficiency. The underlying mechanism in this new criterion is the same as in the Hilbert's infinite Hotel paradox. It will be applied to prove the Pareto inefficiency of equilibrium in an example involving a quadratic regeneration function of the resource, where assumption A3 is not satisfied, making the criterion in Theorem 4 insufficient to guarantee the Pareto inefficiency of the equilibrium.

[15] Rhee (1991) presents a counterexample for an OLG economy with land.

[16] Other forms of utility functions can be considered, but the resulting formulas would be complex and less elegant, so they are not presented here.

4.2 Corollaries

From the above theorems, we immediately have the following corollaries.

Corollary 1. Under the assumptions **A1,A2**, the equilibrium is Pareto-efficient if and only if $\lim_{t\to\infty} D_t V_t = 0$.

Remark 7. Under **A1,A2**, the condition $\lim_{t\to\infty} D_t V_t = 0$ is a complete characterization for Pareto efficiency of the equilibrium.

Corollary 2. If

$$\overline{\lim_{t\to\infty}} \frac{R_{t+1}}{R_t G'(S_t - R_t)} < 1, \tag{8}$$

$$\overline{\lim_{t\to\infty}} \frac{p_t R_t}{\omega_t N_t} > 0, \tag{9}$$

then, the equilibrium is Pareto-efficient.

Remark 8. Condition (8) indicates that for large t,

$$\frac{R_{t+1}}{R_t} < G'(S_t - R_t),$$

which means that the growth speed of harvesting is lower than the marginal regeneration capacity. In other words, the natural resource is not extracted too quickly; condition (9) means that in the long run, in production, the resource share is not nil, compared with the labor share. Condition (9) is weaker than the assumption **A2**.

Corollary 3. If

$$\overline{\lim_{t\to\infty}} \frac{1 + j_t}{1 + r_t} < 1, \tag{10}$$

$$\overline{\lim_{t\to\infty}} \frac{(1 + r_t)K_t}{\omega_t N_t} > 0, \tag{11}$$

then, the equilibrium is Pareto-efficient.

Remark 9. Roughly speaking, condition (10) indicates that the growth rate of capital is less than the interest rate, which can be seen as the reverse inequality of condition (4); condition (11) means that the physical capital is important in production relative to labor. This result can be seen as the converse of Theorem 4 to some extent and also as a modified version of the dividend criterion for Pareto efficiency of AMZS (1989)[17].

[17] See Remark 5.

By combining Theorem 4 and Corollary 3, we can infer that the long-term relationship between the growth rate of capital and the interest rate is crucial in determining economic efficiency. Roughly, if the growth rate of capital consistently exceeds the interest rate in the long run, it may result in over-accumulation of capital, leading to inefficiency. Conversely, if the growth rate of capital remains below the interest rate in the long run, capital accumulation is more controlled, promoting efficiency. This conclusion aligns with our intuitive understanding. In summary, comparing the long-term growth rate of capital to the interest rate provides a meaningful approach to assessing economic efficiency. Of course, for rigorousness, certain preassumptions would need to be made.

Corollary 4. Under assumption **A3**, furthermore, suppose that as $t \to \infty$,

$$\frac{K_t}{B_t N_t} \to k^* > 0, \quad r_t \to r^* > -1, \quad \frac{\omega_t}{B_t} \to \omega^* > 0,$$

where k^*, r^*, ω^* are constants. Then, the equilibrium is Pareto-efficient if $r^* > n'$; it is Pareto-inefficient if $r^* < n'$, where $n' = (1+\nu)(1+n) - 1$ is the growth rate of capital in the long run.

Remark 10. Theorems 1, 3, 4, and 5, and Corollaries 3 and 4 remain valid in the absence of natural resources. This is because the proofs of these results do not rely on the assumption that "the production function at any time is strictly increasing with respect to the natural resource." (See subsection 3.1 The Economy.). However, Theorem 2 does require this assumption for its validity.

5 Applications

In this section, we apply the above general criteria to several examples where the regeneration function is either linear or quadratic.

The quadratic regeneration function is the simplest model for illustrating the characteristic that any renewable resource has a finite environmental carrying capacity. Consequently, it is widely used in resource economics.

In contrast, the linear regeneration function is typically used for nonrenewable resources, resources that degenerate exponentially, or idealized renewable resources with infinite environmental carrying capacity.

In all of these concrete examples, we first need to identify the equilibrium (or equilibria, if multiple exist) and then evaluate its efficiency. This is where our criteria come into play. While finding the equilibrium can be challenging, our primary focus is on assessing its efficiency.

Moreover, none of these examples is trivial; each represents an intriguing model in resource economics, highlighting some important features of resource use. Specifically, there are three models presented here. A simplified version of the first model appeared in Agnani et al. (2005); the other two are original to this paper.

In all examples in this section, we assume that the utility function is as follows: $U(a, b) = \ln a + \rho \ln b$, where $\rho \in (0, 1)$ is a constant.

5.1 Linear Regeneration Function, Cobb–Douglas Production Function

Assume

$$G(x) = \eta x, \quad F^t(K, L, R) = A_t K^\alpha L^\beta R^\gamma, \quad \forall t \in \mathbb{N},$$

where $A_t > 0$, $\eta > 0$, $0 < \alpha, \beta, \gamma < 1$ are constants, satisfying $\alpha + \beta + \gamma = 1$. The parameter η can be called the intensity of the regeneration of the resource.

Agnani et al. (2005) investigate the case of an exhaustible resource where $\eta = 1$. However, they implicitly assume the existence of equilibrium and further assume that the economy follows an exact balanced growth path, with both A_t and N_t growing at exogenously given rates. Under such narrowly defined conditions, they discuss issues of Pareto efficiency, sustainability, and social optimality.

In contrast, we rigorously prove the existence and uniqueness of equilibrium, as well as its Pareto efficiency and social optimality, while also examining the sustainability issue in a more general context.

Mourmouras (1991) demonstrates the existence of equilibrium and addresses sustainability in the absence of technical progress and population change.

Our results in this subsection fully encompass both Agnani et al. (2005) and Mourmouras (1991).

For use in the sequel, let $\theta = \rho\beta/(1+\rho)$, $\delta = 1 - \tau$, and $\tau \in (0,1)$ is determined uniquely by

$$\tau = \frac{\gamma + \alpha\tau}{\theta + \gamma + \alpha\tau}.$$

5.1.1 Equilibrium Existence, Uniqueness and Efficiency

By the definition of equilibrium, one can easily verify that the equilibrium exists and is unique if and only if the following system of equations for $(K_t, S_t, R_t)_{t\in\mathbb{N}}$ with $K_0 = \overline{K}_0$ and $S_0 = \overline{S}_0$ has a unique positive solution:

$$K_{t+1} = \frac{Y_t}{R_t}\left[(\theta+\gamma)R_t - \gamma S_t\right], \tag{12}$$

$$S_{t+1} = \eta(S_t - R_t), \tag{13}$$

$$R_{t+1} = \frac{\eta}{\alpha}\left[(\theta+\gamma)R_t - \gamma S_t\right], \tag{14}$$

where $Y_t = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma}$. And, obviously, this system of equations has a unique positive solution if and only if the planar difference dynamical system for $(S_t, R_t)_{t\in\mathbb{N}}$ with $S_0 = \overline{S}_0$, described by (13) and (14), has a unique positive solution.

Following this line of reasoning, we can derive

Proposition 1. The equilibrium exists and is unique, in which for any $t \in \mathbb{N}$, $N_t a_t = \frac{\beta}{1+\rho} Y_t$, $N_{t-1} b_t = (\alpha + \gamma/\tau) Y_t$, $R_t = \tau S_t$, $K_{t+1} = \alpha\delta Y_t$, where $Y_t = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma}$.

Remark 11. The resource extraction rate is constant τ, and the total consumption of the young, the total consumption of the old, and the investment in capital are all proportional to the total output, each by a constant ratio.

Proposition 2. The equilibrium is Pareto-efficient.

Remark 12. While the existence and uniqueness of equilibrium are expected, the Pareto efficiency result is surprising. This sharply contrasts with the classical Diamond OLG model without natural resources, which can be seen as a special case of our model when $\gamma = 0$.

To explain the sharp contrast between the cases of $\gamma > 0$ and $\gamma = 0$, we compare the growth rate of capital and the interest rate.

Notice that in the equilibrium of this economy, for any $t \in \mathbb{N}$,

$$\frac{1+j_t}{1+r_t} = \frac{K_{t+1}}{(1+r_t)K_t} = \frac{K_{t+1}}{\alpha Y_t} = \delta.$$

This implies that the growth rate of capital is always lower than the interest rate, preventing overaccumulation of capital.

In contrast, in the classical Diamond model, (where assumption **A3** is satisfied), for any $t \in \mathbb{N}$,

$$\frac{1+j_t}{1+r_t} = \frac{\theta}{\alpha}.$$

Therefore, two cases may arise. If $\alpha > \theta$, then, the growth rate of capital is always lower than the interest rate, preventing overaccumulation. However, if $\alpha < \theta$, then the growth rate of physical capital is always higher than the interest rate, leading to capital overaccumulation.

5.1.2 Social Optimality

We make an assumption:

A4. $A_t = (1+g)^t, \quad N_t = (1+n)^t, \quad \forall t \in \mathbb{N}$, where $g \geq 0, n > -1$ are constants.

Proposition 3. Under **A4**, the equilibrium allocation is socially optimal with respect to W_δ.

In the previous subsection, we demonstrated Pareto efficiency without assumption **A4**. With this assumption, however, we can derive a stronger result. Here, δ represents the social discount factor embedded in the market system, whereas ρ corresponds to the individual discount factor.

The social discount factor is determined by the aggregation of individual discount rates across all economic agents in the economy, reflecting the underlying economic structure, including aspects such as resource availability and the relative significance of labor.

Furthermore, δ increases with respect to ρ, indicating a positive correlation between the social and individual discount factors.

Which is larger, δ or ρ? The answer primarily depends on the labor share, β. If β is sufficiently small, then $\delta < \rho$; if β is sufficiently large, then $\delta > \rho$. Specifically, when $\beta = 0$, we have $\delta = 0$, and when $\beta = 1$, we have $\delta = 1$. This suggests that the greater the role of labor in production, the less the "social planner" discounts the future.

Additionally, because $\delta = 1 - \tau$, where τ represents the optimal resource extraction rate in the market, a heavier discounting of the future by the "social planner" implies more rapid resource extraction.

5.1.3 Sustainability

Denote the output per capita at time t as $y_t = Y_t/N_t$. Because

$$Y_{t+1} = A_{t+1}(\alpha\delta Y_t)^\alpha N_{t+1}^\beta R_{t+1}^\gamma,$$

then, $y_{t+1} = m_t y_t^\alpha$, where

$$m_t \sim \frac{A_{t+1}N_t^\alpha}{N_{t+1}^{\alpha+\gamma}}(\eta\delta)^{\gamma t}.$$

Clearly, the behavior of the economy depends on the long-term behavior of m_t. If $\lim\limits_{t\to\infty} m_t = \infty$, the economy grows without bound. If $\lim\limits_{t\to\infty} m_t = m$ for some $m > 0$, the economy converges to a finite level. If $\lim\limits_{t\to\infty} m_t = 0$, the economy contracts, leading to a collapse. If $(m_t)_{t\in\mathbb{N}}$ does not converge, the economy exhibits fluctuations.

In particular, under **A4**, we have

$$m_t \sim h^t, \quad h := (1+g)\left(\frac{\eta\delta}{1+n}\right)^\gamma.$$

Here, h is a composite index, indicating the extent of sustainability. If $h > 1$, the economy expands without bound. If $h = 1$, the economy converges to a finite level. If $h < 1$, the economy contracts.

For example, for a given technical growth rate $g > 0$, if the resource share γ is sufficiently small, then the economy will expand without bound.

For given g, n, η, γ, if the discount factor ρ or the labor share β is sufficiently small, then the resource harvesting rate τ is sufficiently close to 1, and δ is sufficiently low, leading to $h < 1$, and consequently, the economy will contract.

5.2 Linear Regeneration Function, CES Production Function

Now, consider other types of CES functions beyond Cobb-Douglas. More precisely, assume $G(x) = \eta x$, and for any $t \in \mathbb{N}$,

$$F^t(K, L, R) = (\alpha K^\sigma + \beta L^\sigma + \gamma R^\sigma)^{1/\sigma}, \quad N_t = (1+n)^t,$$

where[18] $0 \neq \sigma < 1, \eta > 0, n \geq 0$, $0 < \alpha, \beta, \gamma < 1$ are given constants, satisfying $\alpha + \beta + \gamma = 1$.

To the best of our knowledge, this model in resource economics is novel and has not been covered in existing literature.

Denote the capital, resource stock, and resource extraction per capita respectively as

$$k_t = \frac{K_t}{N_t}, \quad s_t = \frac{S_t}{N_t}, \quad z_t = \frac{R_t}{N_t}.$$

It is easy to see that the equilibrium exists if and only if the following three-dimensional difference dynamical system $\mathscr{D}$ of $(k_t, s_t, z_t)_{t \in \mathbb{N}}$ with given $k_0 > 0, s_0 > 0$ has positive solution:

$$k_{t+1} = \frac{1}{1+n} (\alpha k_t^\sigma + \beta + \gamma z_t^\sigma)^{(1-\sigma)/\sigma} \left[\theta - \frac{\gamma(s_t - z_t)}{z_t^{1-\sigma}}\right], \tag{15}$$

$$s_{t+1} = \frac{\eta}{1+n}(s_t - z_t), \tag{16}$$

$$z_{t+1} = \frac{1}{1+n} \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{z_t}{\alpha k_t^\sigma + \beta + \gamma z_t^\sigma} \left[\theta - \frac{\gamma(s_t - z_t)}{z_t^{1-\sigma}}\right], \tag{17}$$

where $\theta = \rho\beta/(1+\rho)$.

[18] When $\sigma = 0$, it will reduce to the Cobb–Douglas case, which is fully analyzed in subsection 5.1. In this subsection, we focus exclusively on the case where $\sigma \neq 0$, except in subsubsection 5.2.3, where we discuss the role of σ in shaping the behavior of the economy.

Clearly, any positive solution of $\mathscr{D}$ must satisfy the feasibility condition that for any $t \in \mathbb{N}$,

$$(s_t, z_t) \in \Theta := \left\{ (s, z) \middle| 0 < z < s < z + \frac{\theta}{\gamma} z^{1-\sigma} \right\},$$

and the limiting condition: there exist $\hat{k}, \hat{s}, \hat{z} \in [0, \infty]$ such that as $t \to \infty$, $k_t \to \hat{k}$, $s_t \to \hat{s}$, $z_t \to \hat{z}$. All other paths will cross out of the region Θ within a finite time, leading to system collapse.

There are only three possible cases: (i) $\hat{s} = \hat{z} = 0$; (ii) $\hat{s}, \hat{z} \in (0, \infty)$; (iii) $\hat{s} = \hat{z} = \infty$, and correspondingly, the equilibrium is called equilibrium of type I, type II, and type III, respectively. And in any case, there exists a $\epsilon \in [0, \theta]$ such that

$$\lim_{t \to \infty} \frac{s_t - z_t}{z_t^{1-\sigma}} = \frac{\theta - \epsilon}{\gamma}. \tag{18}$$

This ϵ can be interpreted as a measure of the speed of resource harvesting, referred to as the harvesting speed indicator, for short.

For the steady state of $\mathscr{D}$, one can easily obtain the following result, which we present as a lemma. For convenience, let

$$\nu := \frac{\rho}{1+\rho} \left[\left(\frac{\eta^\sigma}{\alpha} \right)^{1/(1-\sigma)} - 1 \right].$$

Clearly, $\nu > (=, <) 1$ if and only if $\alpha < (=, >) \eta^\sigma \left(2 + \rho^{-1}\right)^{\sigma - 1}$.

Lemma 2. If

$$\frac{1+n}{\eta} < \min\{1, \nu\} \tag{19}$$

then, there exists a unique steady state of $\mathscr{D}$, which is a saddle[19]. Otherwise, there is no steady state.

We observe that when $\sigma > 0$, condition (19) implies that η is large, whereas when $\sigma > 0$, condition (19) means that η lies within a certain interval, neither too large nor too small.

Concerning the equilibrium of type II, from Lemma 2, one can easily obtain

[19] It can be verified by the eigenvalue method.

Proposition 4. If the condition (19) is satisfied, the equilibrium exists and is unique, and it is of type II and induced by the unique saddle path of $\mathscr{D}$. Otherwise, no type II equilibrium exists.

Concerning the efficiency, we have

Proposition 5. The type II equilibrium is Pareto-efficient.

Therefore, the steady state equilibrium is always Pareto-efficient if it exists.

In the sequel, we will examine the existence and efficiency of type I and type III equilibria in two distinct cases: $\sigma \in (0,1)$ and $\sigma < 0$. For the sake of completeness, when presenting results regarding the existence of equilibrium, we will also include type II equilibria.

5.2.1 $\sigma \in (0,1)$

For any $k \geq 0$, define

$$\pi(k) = (1+n)k\left(\alpha k^{\sigma} + \beta\right)^{(\sigma-1)/\sigma}.$$

It is easy to verify that π is strictly increasing.

The meaning of π is as follows. For a type I equilibrium, there is a $\epsilon \in [0,\theta]$ such that condition (18) holds, and then, by letting $t \to \infty$ in (15), we obtain that the limiting capital per capita k satisfies

$$k = \frac{\epsilon}{1+n}(\alpha k^{\sigma} + \beta)^{(1-\sigma)/\sigma},$$

which implies $\epsilon = \pi(k)$. Therefore, π represents the one-to-one relationship between the harvesting speed indicator and the limiting capital per capita, and this relationship is positive. In other words, the faster the resource extraction, the higher the limiting capital per capita.

Concerning the existence of equilibrium, we have

Proposition 6. (i) No type III equilibrium exists.

(ii) If $\frac{1+n}{\eta} < \min\{1,\nu\}$, then, there exists unique equilibrium, which is of type II.

(iii) If $\nu \leq \frac{1+n}{\eta} < 1$, then, there exists a unique type I equilibrium with limiting capital per capita being $\pi^{-1}(\theta)$.

(iv) If $\frac{1+n}{\eta} \geq 1$, then, there exists a continuum of type I equilibria. More precisely, there exists an interval $[\underline{z}, \overline{z}]$ such that any $z_0 \in [\underline{z}, \overline{z}]$ induces a type I equilibrium. Accordingly, there exists a $\overline{k} \in (0, \pi^{-1}(\theta)]$ such that for any $k \in [0, \overline{k}]$, there is a type I equilibrium of with the limiting capital per capita being k. The more z_0, the more the limiting capital per capita k.

Remark 13. If the regeneration capacity is relatively large compared with the population growth rate and there is no steady state, then, there is a unique equilibrium in which the resource is harvested as quickly as possible (the harvesting speed indicator is θ) and finally the resource stock tends to zero, and correspondingly, the capital per capita tends to the largest possible value $\pi^{-1}(\theta)$. If the regeneration capacity is relatively small compared with the population growth rate, then, there is a continuum of equilibria.

Concerning the efficiency of equilibrium, we have

Proposition 7. (i) If $\eta > 1 + n$, then the unique equilibrium is Pareto-efficient.

(ii) If $\eta \leq 1 + n$, then, there exists a $z_* \in [\underline{z}, \overline{z}]$ such that for any $z_0 \in [\underline{z}, z_*)$, the corresponding equilibrium of type I is Pareto-efficient; for any $z_0 \in (z_*, \overline{z}]$, the corresponding equilibrium of type I is Pareto-inefficient. Accordingly, there exists a $\underline{k} \in (0, \overline{k}]$ such that for any $k \in [0, \underline{k})$, the corresponding equilibrium of type I is Pareto-efficient; for any $k \in (\underline{k}, \overline{k}]$, the corresponding equilibrium of type I is Pareto-inefficient.

Remark 14. This implies that if the regeneration capacity is relatively large compared with the population growth rate, then, the unique equilibrium is Pareto-efficient.

If the regeneration capacity is relatively small compared with the population growth rate, the resource stock per capita will tend to zero, and there exists a threshold for the initial resource extraction z_0 (accordingly, there exist a threshold for the harvesting speed indicator ϵ and a threshold for limiting capital per capita k), below which the economy is Pareto-efficient and above which it becomes inefficient. In other words, the slower the resource extraction, the higher the likelihood that the economy is Pareto-efficient.

5.2.2 $\sigma < 0$

Denote

$$\epsilon_* := (1+n)\beta(1-\sigma^{-1})\left[\alpha(1-\sigma)\right]^{-1/\sigma},$$
$$\alpha^* := (1-\sigma)^{-1}\left[(1+n)(1-\sigma^{-1})\frac{1+\rho}{\rho}\right]^{\sigma}.$$

Clearly, $\theta > (=,<)\epsilon_*$ if and only if $\alpha < (=,>)\alpha^*$.

Concerning the existence of equilibrium, we have

Proposition 8. (i) If $\frac{1+n}{\eta} \geq 1$, then, there exists a continuum of equilibria of type I, each with zero limiting capital per capita.

(ii) If $\frac{1+n}{\eta} < \min\{1,\nu\}$, then, there exists a unique equilibrium, which is of type II.

(iii) If $\nu \leq \frac{1+n}{\eta} < 1$, then if

$$\alpha = \alpha^*, \quad \frac{1+n}{\eta} \leq -\frac{\sigma\rho}{1+\rho}, \tag{20}$$

or

$$\alpha < \alpha^*, \quad \nu = \frac{1+n}{\eta} > -\frac{\sigma\rho}{1+\rho}, \tag{21}$$

then there exists a unique type III equilibrium with harvesting speed indicator θ; if

$$\alpha < \alpha^*, \quad \nu \leq \frac{1+n}{\eta} < -\frac{\sigma\rho}{1+\rho}, \tag{22}$$

then there exists a continuum of type III equilibria. More precisely, there is a $\epsilon^* \in [\epsilon_*, \theta)$ such that for any $\epsilon \in [\epsilon^*, \theta]$, there is a type III equilibrium with harvesting speed indicator ϵ. In all other cases, no equilibrium exists.

Concerning the efficiency of equilibrium, we have

Proposition 9. (i) If $\eta \leq 1+n$, then, any type I equilibrium is Pareto-efficient.

(ii) If $\nu \leq \frac{1+n}{\eta} < 1$, then, there are three cases.

First, if (20) holds, then, the unique type III equilibrium is Pareto-efficient if $-\sigma < \frac{1+\rho}{\rho}$; it is Pareto-inefficient if $-\sigma > \frac{1+\rho}{\rho}$.

Second, if (21) holds, then, the unique type III equilibrium is Pareto-efficient.

Third, if (22) holds, then, there exists $\bar{\epsilon} \in [\epsilon^*, \theta]$ such that the type III equilibrium with harvesting speed indicator below $\bar{\epsilon}$ is Pareto-efficient; whereas any type III equilibrium with harvesting speed indicator above $\bar{\epsilon}$ is Pareto-inefficient.

Remark 15. When $\sigma < 0$, the natural resource is essential for production, meaning that output drops to zero in its absence.

If the regeneration capacity of the resource is weak compared to the population growth, the resource stock per capita will tend to zero, making over-accumulation of the resource impossible. Additionally, because the factors are complementary, capital and the resource are closely linked in production, preventing over-accumulation of capital and thereby ensuring the Pareto efficiency of the economy.

If the regeneration capacity of the resource is strong compared to the population growth and there is no steady state, the resource stock per capita will tend to infinity. And, in general, similar to the case when $\sigma \in (0, 1)$, there exists a threshold for the harvesting speed indicator, below which the economy is Pareto-efficient, and above which it becomes inefficient. In other words, the slower the resource extraction, the higher the likelihood that the economy will be Pareto-efficient. And this threshold, in principle, is concerned with the marginal regeneration capacity.

Remark 16. Regarding the nonexistence of equilibrium. Typically, when $\nu < \frac{1+n}{\eta} < 1$, and $\alpha > \alpha^*$, no equilibrium exists. This suggests that in a typical scenario where the regeneration capacity of the resource is very strong and the capital share excessively large, but the resource and the capital are complementary in production, they cannot maintain a coherent relationship, preventing the economy from following an equilibrium path.

5.2.3 Comparison Between Different σ

We know that $\sigma \in (0, 1)$ indicates substitutability between the factors, $\sigma < 0$ indicates complementarity between the factors, and $\sigma = 0$ represents the midpoint between the two.

Here, we compare the behavior of the dynamical system $\mathscr{D}$ under different values of σ, including $\sigma = 0$. Specifically, we examine how varying σ influences the system's trajectories, equilibrium types, and overall economic efficiency. By contrasting these cases, we gain insight into the role that σ plays in determining the system's stability and optimality.

In the Cobb–Douglas case where $\sigma = 0$, according to (12), (13) and (14), the dynamical system $\mathscr{D}$ simplifies to

$$\begin{aligned}
k_{t+1} &= \frac{1}{1+n} k_t^{\alpha} z_t^{\gamma-1} \left[(\theta+\gamma) z_t - \gamma s_t\right], \\
s_{t+1} &= \frac{\eta}{1+n}(s_t - z_t), \\
z_{t+1} &= \frac{\eta}{\alpha(1+n)} \left[(\theta+\gamma) z_t - \gamma s_t\right].
\end{aligned}$$

From the analysis of the Cobb–Douglas case in subsection 5.1, we know that when $\sigma = 0$, this dynamical system has a unique positive solution, and the economy possesses a unique equilibrium, which can be of type I, type II, or type III, depending on whether $\eta\delta <, =, > 1+n$, respectively, and it is socially optimal with respect to W_δ, and therefore Pareto-efficient.

For the dynamical system $\mathscr{D}$, including the case $\sigma = 0$, the system behavior changes as σ varies. Depending on other parameters, the system may exhibit continuity in some cases, whereas in others, bifurcation may occur.

First of all, regarding the steady state, we know that for $\sigma \neq 0$, the steady state exists if and only if condition (18) holds, and for $\sigma = 0$, the steady state exists if and only if $\eta\delta = 1+n$. For $\sigma = 0$, (18) reduces to

$$\frac{1+n}{\eta} < \min\left\{1, \frac{\rho}{1+\rho}\left(\frac{1}{\alpha} - 1\right)\right\},$$

which is satisfied naturally when $\eta\delta = 1+n$. In fact, noticing $\delta \in (0,1)$, we only need to verify

$$\delta < \frac{\theta}{\beta}\frac{\beta+\gamma}{\alpha},$$

which follows from

$$\frac{\alpha}{\theta+\gamma+\alpha\tau} = \frac{\alpha\tau}{\gamma+\alpha\tau} < \frac{\beta+\gamma}{\beta}.$$

Therefore, regarding the steady state, the dynamical system $\mathscr{D}$ exhibits continuity with respect to σ.

In the sequel, we present some examples to further illustrate the role of σ in the system's behavior.

Example 1. Suppose $\alpha < 1/(2e)$, and

$$\frac{\rho}{1+\rho}\frac{\beta+\gamma}{\alpha} < \frac{1+n}{\eta} < \delta.$$

For $\sigma = 0$, because $\eta\delta > 1+n$, then, there exists a unique trajectory of $\mathscr{D}$, along which each of k_t, s_t, z_t tends to infinity, all other trajectories lead to system collapse within a finite time, and correspondingly, there is a unique equilibrium of type III, which is of course Pareto-efficient.

For $\sigma > 0$ near $\sigma = 0$ locally, becuse $\nu < \frac{1+n}{\eta} < 1$, then, there exists a unique trajectory of $\mathscr{D}$, along which (k_t, s_t, z_t) tends to $(\pi^{-1}(\theta), 0, 0)$, all other trajectories lead to system collapse within a finite time, and correspondingly, there is a unique equilibrium of type I, which is Pareto-efficient.

For $\sigma < 0$ near $\sigma = 0$ locally, we have $\nu < \frac{1+n}{\eta} < 1$. In addition, locally near $\sigma = 0$, α^* is sufficiently close to $1/e$, and thus, locally near $\sigma = 0$, we have $\alpha < \alpha^*$. But, locally near $\sigma = 0$, it does not hold that

$$\frac{1+n}{\eta} > -\frac{\sigma\rho}{1+\rho}.$$

Therefore, by Proposition 8, there is no equilibrium. All trajectories of $\mathscr{D}$ lead to system collapse within a finite time.

Thus, in this case, bifurcation occurs at $\sigma = 0$.

Example 2. Suppose $\eta < 1+n$. Then, locally near $\sigma = 0$, the system exhibits continuity. In fact, regardless of the value of σ, $\mathscr{D}$ has trajectories (possibly unique) converging to $(k, 0, 0)$ for some (different) k, all other trajectories exit the region Θ within a finite time. Correspondingly, for any σ, the economy consistently exhibits type I equilibria. The efficiency of the equilibria manifests differently: for $\sigma \in (0, 1)$, there exists a threshold for the harvesting speed indicator, below which the economy is Pareto-efficient, and above which it becomes inefficient; for $\sigma \leq 0$, all equilibria are Pareto-efficient.

To sum up, roughly, if $\eta > 1+n$, then the system may exhibit bifurcation at $\sigma = 0$; on the contrary, if $\eta < 1+n$, then the system exhibits continuity at $\sigma = 0$.

Additionally, in general, in most cases, the economy exhibits multiple equilibria. However, in the Cobb-Douglas case, where all variables are in fixed proportion, the set of equilibria reduces to a singleton.

The primary distinction between the cases $\sigma > 0$ and $\sigma < 0$ lies in the behavior of the resource stock per capita. When $\sigma > 0$, the resource stock per capita cannot tend to infinity and thus there is no type III equilibrium, even if the resource regeneration capacity is very large. However, when $\sigma < 0$, this becomes possible. Additionally, when $\sigma < 0$, the possibility of nonequilibrium arises.

A key commonality between the cases $\sigma \in (0,1)$ and $\sigma < 0$ is that, roughly speaking, the slower the resource extraction, the higher the likelihood that the economy will be Pareto-efficient. This principle holds regardless of whether the factors are substitutable or complementary. But, more precisely, this principle applies to the case $\eta < 1+n$ when $\sigma \in (0,1)$ (the resource stock per capita tends to zero) and to the case $\eta > 1+n$ when $\sigma < 0$ (the resource stock per capita tends to infinity). The intuition behind this principle, within the framework of general CES technology, is that whether the factors are substitutable or complementary, they remain interconnected through a weak proportional relationship. Faster resource extraction leads to greater resource use in production, which increases the demand for capital in the production process. Over time, this results in higher capital accumulation, which raises the likelihood of capital overaccumulation, potentially leading to inefficiency.

5.2.4 Comparison with Classical Diamond OLG Model

Our model in the general CES form reduces to the classical Diamond OLG model without natural resources when $\gamma = 0$.

It's easy to see that in this Diamond model, there exists a unique equilibrium, in which for any $t \in \mathbb{N}$,

$$\begin{aligned}
a_t &= \frac{\beta}{1+\rho}\left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}, \\
\frac{b_t}{1+n} &= \alpha k_t^\sigma \left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}, \\
(1+n)k_{t+1} &= \theta\left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}, \\
\omega_t &= \beta\left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}, \\
1 + r_t &= \alpha k_t^{\sigma-1}\left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}.
\end{aligned}$$

In the sequel, we only consider the case, where $\sigma < 0$. Define

$$n^* =: \frac{\rho\left[\alpha(1-\sigma)\right]^{1/\sigma}}{\left(1-\sigma^{-1}\right)(1+\rho)} - 1, \quad n_* =: \left[\alpha\left(1 + \frac{1+\rho}{\rho}\right)^{1-\sigma}\right]^{1/\sigma} - 1.$$

One can check directly that $-1 < n_* < n^*$.

It's easy to verify that if $n > n^*$, then, $\lim_{t\to\infty} k_t = 0$. Therefore, as $t \to \infty$,

$$\frac{1}{1+r_{t+1}}\frac{\omega_{t+1}N_{t+1}}{\omega_t N_t} = \frac{1+n}{\alpha}\frac{k_{t+1}^{1-\sigma}}{\left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}} = \frac{\theta}{\alpha}k_{t+1}^{-\sigma} \to 0,$$

which implies $\lim_{t\to\infty} D_t \omega_t N_t = 0$. Then, by Theorem 3, the equilibrium is Pareto-efficient.

In the following, suppose $n < n^*$. Then, the limiting capital per capita k and the limiting interest rate r satisfy

$$\frac{1+r}{1+n} = x = \frac{\alpha}{\theta}k^\sigma,$$

and x is the smaller one of the two positive roots of the equation

$$(1+n)^\sigma x = \alpha\left(x + \beta/\theta\right)^{1-\sigma}.$$

If $n \in (n_*, n^*)$, then, one can check that $x > 1$. Therefore, $r > n$. In addition, the limiting wage is positive. Therefore, by Corollary 4, the equilibrium is Pareto-efficient.

If $n < n_*$, then, one can check that $x < 1$. Therefore, $r < n$. In addition, the limiting wage is positive. Therefore, by Corollary 4, the equilibrium is Pareto-inefficient.

However, in the case $n < n_*$, when natural resources, as in our CES model, are introduced into the economy, the economy becomes Pareto-efficient, provided that $\eta < 1 + n$.

Thus, we can conclude that in certain cases, the presence of natural resources can enhance the efficiency of the economy, especially when the capital and the resource are complementary in production.

At the end of this subsection, we point out that the sustainability issue in this CES model is straightforward and thus omitted from the discussion.

5.3 Quadratic Regeneration Function

Assume $G(x) = \lambda x(1 - x/B)$, and for any $t \in \mathbb{N}$,

$$F^t(K, L, R) = A_t K^\alpha L^\beta R^\gamma, \quad N_t = (1+n)^t, \quad A_t = (1+g)^t,$$

where $n \geq 0, g \geq 0, \lambda > 0, B > \overline{S}_0$, and $\rho, \alpha, \beta, \gamma \in (0, 1)$ are constants, satisfying $\alpha + \beta + \gamma = 1$.

And assume λ and B are sufficiently large. Here, λ is the intrinsic growth rate of the natural resource, and B is the environmental carrying capacity for this natural resource.

As mentioned in the literature review, Krautkraemer (1999), in an OLG economy with a natural resource but no physical capital, suggests that when the resource's output share is relatively small, steady-state equilibrium is Pareto-inefficient, but does not specify how small it must be. Here, for an economy with capital, we provide a similar but more precise result regarding the dynamic equilibrium.

5.3.1 Existence and Uniqueness of Equilibrium

It's easy to verify that an equilibrium exists if and only if the following difference dynamical system for $(K_t, S_t, R_t)_{t\in\mathbb{N}}$ with $K_0 = \overline{K}_0$ and $S_0 = \overline{S}_0$ has a positive solution:

$$
\begin{aligned}
K_{t+1} &= \frac{Y_t}{1+\rho}\left[\rho\beta - \left(\frac{G(S_t-R_t)}{G'(S_t-R_t)} + \rho(S_t-R_t)\right)\frac{\gamma}{R_t}\right], && (23)\\
S_{t+1} &= G(S_t-R_t), && (24)\\
R_{t+1} &= \frac{R_t G'(S_t-R_t)}{\alpha(1+\rho)}\left[\rho\beta - \left(\frac{G(S_t-R_t)}{G'(S_t-R_t)} + \rho(S_t-R_t)\right)\frac{\gamma}{R_t}\right], && (25)
\end{aligned}
$$

which, in turn, if and only if the planar difference dynamical system for $(S_t, R_t)_{t\in\mathbb{N}}$, described by (24) and (25), with $S_0 = \overline{S}_0$, has a positive solution.

This planar dynamical system has two steady states: $(0,0)$ and (S^*, R^*), satisfying

$$R^* = G(x^*) - x^*, \quad S^* = G(x^*),$$

where $x^* \in (0, B/2)$ is determined uniquely by

$$\lambda\left(\alpha + \frac{\gamma}{1+\rho}\right)\left(1-\frac{x^*}{B}\right) - \alpha = \frac{\rho\beta\lambda}{1+\rho}\left(1-\frac{x^*}{B/2}\right)\left[\lambda\left(1-\frac{x^*}{B}\right) - 1 - \frac{\gamma}{\beta}\right].$$

By the eigenvalue method[20], one can see that $(0,0)$ is a source, to which no feasible path converges; (S^*, R^*) is a saddle, to which a unique saddle path converges.

Therefore, there exists a unique $R_0 > 0$ which induces a unique path converging to this saddle. The unique equilibrium then follows.

Consequently, we obtain

Proposition 10. The equilibrium exists and is unique, and the corresponding path of $(S_t, R_t)_{t\in\mathbb{N}}$ converges to a saddle.

5.3.2 Pareto Efficiency

Let

$$\kappa = \frac{1+\rho}{\rho}\alpha + \left(\frac{1}{\rho} + \frac{2(1+\rho)}{\rho(\lambda-1)}\right)\gamma,$$

which represents a weighted sum of the two capital shares: α and γ, and hence, can be referred to as a combined capital index. Note that this index only

[20]The Jacobian matrix at the steady state $(0,0)$ has two eigenvalues bigger than 1. Therefore, the steady state $(0,0)$ is a source. The Jacobian matrix at the steady state (S^*, R^*) has two positive eigenvalues: one is smaller than 1, the other is bigger than 1. Therefore, (S^*, R^*) is a saddle.

concerns the intrinsic growth rate of the natural resource, not the carrying capacity.

Proposition 11[21]. The equilibrium is Pareto-efficient if $\beta < \kappa$; it is Pareto-inefficient if $\beta > \kappa$.

Remark 17. The relative magnitude of the labor share plays a crucial role. If the labor share is less than the combined capital index, the equilibrium is efficient; however, if the labor share exceeds the combined capital index, the equilibrium becomes inefficient.

On the simplex $\mathcal{A} = \{(\alpha, \beta, \gamma) | \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \in [0, 1]\}$, the line segment $\beta = \kappa$ has two endpoints, the coordinates of which are $(\overline{\alpha}, \overline{\beta}, 0)$ and $(0, \underline{\beta}, \underline{\gamma})$, respectively, where

$$\overline{\alpha} = \frac{\rho}{1 + 2\rho}, \quad \overline{\beta} = \frac{1 + \rho}{1 + 2\rho}, \quad \underline{\beta} = \frac{1 + \lambda + 2\rho}{(1 + \rho)(1 + \lambda)}, \quad \underline{\gamma} = \frac{\rho(\lambda - 1)}{(1 + \rho)(1 + \lambda)}.$$

Recall λ is sufficiently large, then $\underline{\beta} < \overline{\beta}$.

Clearly, if $\beta < \underline{\beta}$, then $\beta < \kappa$; if $\beta > \underline{\beta}$, then $\beta > \kappa$. And, of course, if $\gamma > \underline{\gamma}$, then $\beta < \underline{\beta}$. Then, from Proposition 9, we can easily get

Corollary 5. If $\beta < \underline{\beta}$, then the equilibrium is Pareto-efficient; if $\beta > \overline{\beta}$, then the equilibrium is Pareto-inefficient.

From this corollary, we can say roughly that if the technology is capital-intensive (either the physical capital or the natural capital), then the economy is efficient; on the contrary, if the technology is labor-intensive, then, the economy is inefficient.

In particular, suppose there is neither technical growth nor population growth. Then K_t will converge to some $K^* > 0$. When $\beta = \kappa$, then $K^* = K_{GR}$, where K_{GR} is the so-called Golden Rule level of capital; when $\beta < \kappa$, then $K^* < K_{GR}$, and the economy is efficient; when $\beta > \kappa$, then $K^* > K_{GR}$, indicating capital overaccumulation, and the economy is inefficient.

[21] If $\gamma = 0$, then, this result coincides with that in the classical Diamond OLG model.

5.3.3 Sustainability

It follows from (23) that

$$y_{t+1} \sim \left(\frac{1+g}{(1+n)^\gamma}\right)^t y_t^\alpha,$$

where $y_y = Y_t/N_t$ is the output per capita. Based on this, we can immediately derive the following result.

Proposition 12. If $1+g < (1+n)^\gamma$, then the economy contracts; if $1+g = (1+n)^\gamma$, then the economy is sustainable in the long run; if $1+g > (1+n)^\gamma$, then the economy grows without bound.

Remark 18. Whether the economy contracts depends solely on the rate of technical progress, population growth, and the resource share. It is independent of the distribution between capital and labor shares.

6 Extension and Further Discussion

6.1 Multiple Resources

The main results can be easily extended to the case of multiple natural resources, where the regeneration capacities are independent of each other. In other words, no cross-effects are present in their regeneration. More specifically, the scenario is as follows.

Consider multiple types of natural resources, labeled type-1, type-2,...,type-J, where J is some natural number.

For any $j = 1, ..., J$, the regeneration function of type-j resource is G_j, being smooth, concave, and nonnegative, defined on $[0, \infty)$, with properties $G_j(0) = 0$, $G'_j(0) \in (0, \infty]$, $G'_j(x) > 0$, $\forall x > 0$.

The dynamics of type-j resource are

$$S^j_{t+1} = G_j(S^j_t - R^j_t),$$

where S^j and R^j represent the stock and extraction of type-j resource, respectively.

In this context, the total value of assets at time t becomes

$$V_t = (1 + r_t)K_t + \sum_{j=1}^{J} p_t^j S_t^j,$$

where p_t^j is the price of type-j resource at time t. The generalized Hotelling rule holds for every type of resources. That is, for any type-j resource, we have

$$\frac{p_{t+1}^j G_j'(S_t^j - R_t^j)}{p_t^j} = 1 + r_{t+1}, \quad \forall t \in \mathbb{N}.$$

In this case, the assumption **A2** "the resource is important in production relative to labor" should be modified to the condition "at least one of the resources is important in production relative to labor."

6.2 OLG with Land

Regarding the OLG economy with land, similar results hold true. Now, the production function is $F^t(K, L, X)$, where X is the input of land.

In an equilibrium, let p_t and q_t be the corresponding price of land and the rental of land at time t, respectively.

The no-arbitrage condition implies that for any $t \in \mathbb{N}$,

$$p_t = \frac{p_{t+1} + q_{t+1}}{1 + r_{t+1}},$$

then, $D_t p_t = D_{t+1} p_{t+1} + D_{t+1} q_{t+1}$. Therefore, $\lim_{t\to\infty} D_t q_t = 0$, and $(D_t p_t)_{t\in\mathbb{N}}$ is decreasing, and hence, there is $\beta_0 \geq 0$ such that $\lim_{t\to\infty} D_t p_t = \beta_0$, and for any $t \in \mathbb{N}$, $p_t = f_t + \beta_t$, where

$$f_t = \frac{1}{D_t} \sum_{s=t+1}^{\infty} D_s q_s, \quad \beta_t = \frac{\beta_0}{D_t},$$

are the fundamental and the bubble of land at t, respectively.

The total value of assets is $V_t = (1 + r_t)K_t + p_t + q_t$, assuming that the quantity of land is one unit. And the total income I_t coincides with wages $\omega_t N_t$.

Assumption **A1** is not needed. In principle, Theorems 1–5 still hold. As a corollary of Theorem 3, if

$$\limsup_{t\to\infty} \frac{q_t}{\omega_t N_t} > 0, \tag{26}$$

that is, land is important in production relative to labor, then $\varliminf_{t\to\infty} D_t\omega_t N_t = 0$. Therefore, the equilibrium is Pareto-efficient.

Condition (26) is a bit weaker than the condition

$$\liminf_{t\to\infty} \frac{q_t}{Y_t} > 0,$$

which is used in Proposition 1 in Rhee (1991) to guarantee the Pareto efficiency of the equilibrium.

6.3 OLG Without Capital

An economy without capital can be viewed as a special case of the general economy with capital, as discussed in Section 3. In this scenario, capital remains at zero throughout, including the initial capital endowment of ancestors. However, the interest rate at any time $t \geq 1$ still exists as a reference for agents borrowing or lending at time $t-1$, though in equilibrium, the quantity of borrowing or lending is zero. In the definition of equilibrium, the interest rate at time $t = 0$ can be ignored because it has no impact on anyone. The generalized Hotelling rule (1) still holds.

From this perspective, all the main results from Sections 3 and 4 continue to hold, except for Theorem 4, where the growth rate of capital is not well defined and thus does not apply in this case.

As an application of Theorem 3, consider an example as follows. Suppose the production functions are of the form: $F^t(L, R) = A_t L f(R/L)$, where $A_t > 0$ is a constant and f is smooth, concave, and satisfying $f(0) = 0$, $f'(0+) \in (0, \infty]$, and the regeneration function is linear: $G(x) = \eta x$, where $\eta > 0$ is a constant. The utility function satisfies the standard conditions such as being smooth, concave, and meeting Inada conditions, etc. Then, analogously to Olson (1997), one can show that the equilibrium exists (possibly multiple).

Take anyone of the equilibria. Since $\eta^{-(t+1)} S_{t+1} = \eta^{-t}(S_t - R_t)$, we have $\sum_{t=0}^{\infty} \eta^{-t} R_t \leq S_0$, therefore $\lim_{t\to\infty} \eta^{-t} R_t = 0$.

Denote the resource extraction and resource stock per capita as $z_t =: R_t/N_t$ and $s_t =: S_t/N_t$, respectively.

Suppose

$$0 < \varliminf_{t\to\infty} \eta^{-t} N_t \le \varlimsup_{t\to\infty} \eta^{-t} N_t < \infty. \tag{27}$$

Then, $\lim_{t\to\infty} z_t = 0$.

In addition, since $F^t(N_t, R_t) = N_t a_t + N_{t-1} b_t \ge N_{t-1} b_t = p_t S_t$, and $p_t = A_t f'(z_t)$, then $f(z_t) \ge f'(z_t) s_t$. By letting $t \to \infty$, and noticing $f(0) = 0$, $f'(0+) \in (0, \infty]$, we obtain $\lim_{t\to\infty} s_t = 0$. That is, along any equilibrium path, the resource stock per capita converges to zero.

Now we consider the efficiency. By the generalized Hotelling rule, for any $t \in \mathbb{N}$, $1 + r_{t+1} = \eta p_{t+1}/p_t$, then, $D_t = \eta^{-t} p_0/p_t$. Therefore, as $t \to \infty$,

$$D_t Y_t = \eta^{-t} \frac{p_0}{A_t f'(z_t)} A_t N_t f(z_t) = p_0 \eta^{-t} N_t \frac{f(z_t)}{f'(z_t)} \to 0,$$

which yields $D_t \omega_t N_t \to 0$. Then, by Theorem 3, the equilibrium is Pareto-efficient.

In some special cases, the condition (27) is superfluous. For example, consider a special case of the example in section 5.1, where $\alpha = 0$, $\overline{K}_0 = 0$. In this case, the production function at time t is $F^t(L, R) = A_t L^\beta R^\gamma$.

In this case, Propositions 1, 2, and 3 still hold. The equilibrium exists and is unique and is Pareto-efficient, in which for any $t \in \mathbb{N}$,

$$N_t a_t = \frac{\beta}{1+\rho} Y_t, \quad N_{t-1} b_t = (\theta + \gamma) Y_t, \quad R_t = \tau S_t,$$

$$1 + r_{t+1} = \frac{Y_{t+1}}{\delta Y_t},$$

where $Y_t = A_t N_t^\beta R_t^\gamma$, $\tau = \gamma/(\theta + \gamma)$, $\delta = 1 - \tau$, $\theta = \rho\beta/(1+\rho)$.

And, under assumption **A4**, the equilibrium allocation is socially optimal with respect to the social welfare functional W_δ. That is, the equilibrium allocation is the solution of the following social planner's problem:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \delta^t \left(\delta \ln a_t + \rho \ln b_t \right), \\
\text{s.t} \quad & S_{t+1} = \eta (S_t - R_t), \\
& N_t a_t + N_{t-1} b_t \le A_t N_t^\beta R_t^\gamma.
\end{aligned}$$

According to Olson (1997): "OLG equilibria differ substantially from the outcome under a social planning exercise, and there does not exist a definitive relation between extractions and prices in the two cases"(290). However, this assertion is incorrect. While the equilibrium allocation may not be socially optimal with respect to W_ρ, which is considered in Olson(1997), it is indeed socially optimal with respect to W_δ. When evaluating social optimality, the individual discount rate should be replaced by the social discount rate, which is embedded in the market system.

7 Conclusion

In this paper, we consider a two-period OLG model with three factors of production: physical capital, labor, and natural resources. We discuss the issue of Pareto efficiency of the equilibrium allocation.

Our main contribution to the literature is that we present general sufficiency conditions and general necessary conditions for the Pareto efficiency of the equilibrium allocation in the OLG economies with natural resources and physical capital. In principle, we compare the growth rates of capital, income, or total asset value with the interest rate. Our findings suggest that, broadly speaking, if any of these growth rates is lower than the interest rate, the equilibrium is efficient. Conversely, if any of these growth rates surpasses the interest rate, the equilibrium becomes inefficient.

A secondary contribution is the finding that, in the case where the resource regeneration function is linear and the production function follows a CES form beyond Cobb–Douglas, there is generally a threshold for the resource harvesting speed. If the harvesting speed is below this threshold, the economy operates efficiently; if it exceeds the threshold, inefficiency arises.

Another contribution is for the case where the resource regeneration function is quadratic. We provide a precise combined capital index and demonstrate that if the labor share is below this index, the economy operates efficiently, whereas if the labor share exceeds this index, the economy becomes inefficient.

Moreover, our findings suggest that under certain conditions, natural resources can enhance economic efficiency, particularly when they are complementary to physical capital in production. This highlights the potential role of resource management in improving economic outcomes.

While our results offer important insights, they remain incomplete. We have not provided general necessary and sufficient conditions for Pareto efficiency of equilibrium, leaving this as an open problem for future research.

Another promising direction for future work is to explore stochastic OLG models that account for uncertainties arising from the random variability of natural resources and environmental conditions. Furthermore, examining government or institutional interventions could be critical, particularly in cases where resource use leads to pollution that exacerbates market inefficiencies.

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Appendix

We need the following three lemmas. The proofs of Lemma 3 and Lemma 4 are straightforward and hence omitted. One proof of Lemma 5 can be found in Farmer et al. (2010), which uses the eigenvalues method in the planar difference dynamical system. Here, we present another proof, which has its own interest.

Lemma 3. A program $\{C_t^*, K_t^*, S_t^*, R_t^*\}_{t\in\mathbb{N}}$ is dynamically efficient if and only if for any program $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$,

$$C_t \geq C_t^*, \quad \forall t \geq 1$$

implies

$$C_0^* \geq C_0.$$

Lemma 4. An allocation $\{a_t^*, b_t^*, K_t^*, S_t^*, R_t^*\}_{t\in\mathbb{N}}$ is Pareto efficient if and only if for any allocation $\{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$,

$$U(a_t, b_{t+1}) \geq U(a_t^*, b_{t+1}^*), \quad \forall t \geq 0$$

implies

$$b_0^* \geq b_0.$$

Lemma 5. The following two statements about $\{S_t, R_t\}_{t\in\mathbb{N}}$ with given $S_0 > 0$ are equivalent:

(I) for any $t \in \mathbb{N}$,

$$\begin{aligned} S_{t+1} &= \eta(S_t - R_t) \geq 0, \\ R_{t+1} &= \frac{\eta}{\alpha}\left[(\theta+\gamma)R_t - \gamma S_t\right] \geq 0; \end{aligned}$$

(II) for any $t \in \mathbb{N}$,

$$\begin{aligned} S_t &= (\eta\delta)^t S_0, \\ R_t &= \tau(\eta\delta)^t S_0, \end{aligned}$$

where δ, τ are defined in the beginning of subsection 5.1.

Proof of Lemma 5. One can easily verify that (II) implies (I). In the sequel, we prove that (I) implies (II). First of all, we show that for any $t \in \mathbb{N}$ and any $n \in \mathbb{N}$, it holds that

$$x_n S_t \leq R_t \leq y_n S_t, \tag{28}$$

where

$$x_{n+1} = \frac{\gamma + \alpha x_n}{\theta + \gamma + \alpha x_n}, \quad x_0 = 0,$$

$$y_{n+1} = \frac{\gamma + \alpha y_n}{\theta + \gamma + \alpha y_n}, \quad y_0 = 1.$$

We prove (28) by use of the method of mathematical induction with respect to n. First, obviously, (28) holds for $n = 0$ and any $t \in \mathbb{N}$. Now, suppose that (28) holds for n and any $t \in \mathbb{N}$. Then, for any $t \in \mathbb{N}$, notice that (28) holds for n and $t + 1$, that is,

$$x_n S_{t+1} \leq R_{t+1} \leq y_n S_{t+1},$$

which is equivalent to

$$x_{n+1} S_t \leq R_t \leq y_{n+1} S_t,$$

and hence, (28) also holds for $n + 1$ and any $t \in \mathbb{N}$. It follows that (28) holds for any $t \in \mathbb{N}$ and any $n \in \mathbb{N}$.

Next, clearly, $\{x_n\}_{n\in\mathbb{N}}$ is increasing and bounded above, and $\{y_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below, and hence, each of these two sequences has a limit. Let $\lim\limits_{t\to\infty} x_t = x$, $\lim\limits_{t\to\infty} y_t = y$. Then, $x, y \in (0, 1)$ and satisfy

$$x = \frac{\gamma + \alpha x}{\theta + \gamma + \alpha x}, \quad y = \frac{\gamma + \alpha y}{\theta + \gamma + \alpha y},$$

which implies $x = y = \tau$. Consequently, $R_t = \tau S_t$, $\forall t \in \mathbb{N}$, which yields (I) immediately. The proof is completed.

Proof of Theorem 1. By Lemma 3, it suffices to show that for any program $\{C'_t, K'_t, S'_t, R'_t\}_{t\in\mathbb{N}}$, if $C'_t \geq C_t$, $\forall t \geq 1$, then $C'_0 \leq C_0$.

In fact, taking large $T \in \mathbb{N}$ arbitrarily, we have

$$\begin{aligned}
& C_0' - C_0 \\
\leq \; & \sum_{t=0}^{T-1} D_t \left[\left(F^t(K_t', N_t, R_t') - K_{t+1}' \right) - \left(F^t(K_t, N_t, R_t) - K_{t+1} \right) \right] \\
& + \sum_{t=0}^{T-1} D_{t+1} p_{t+1} \left[\left(G(S_t' - R_t') - S_{t+1}' \right) - \left(G(S_t - R_t) - S_{t+1} \right) \right] \\
\leq \; & \sum_{t=0}^{T-1} D_t \left[(1 + r_t)(K_t' - K_t) + p_t (R_t' - R_t) - (K_{t+1}' - K_{t+1}) \right] \\
& + \sum_{t=0}^{T-1} D_{t+1} p_{t+1} \left\{ G'(S_t - R_t) \left[(S_t' - S_t) - (R_t' - R_t) \right] - \left(S_{t+1}' - S_{t+1} \right) \right\} \\
= \; & \sum_{t=0}^{T-1} \left[D_{t-1}(K_t' - K_t) + D_t p_t (R_t' - R_t) \right] - \sum_{t=1}^{T} D_{t-1}(K_t' - K_t) \\
& + \sum_{t=0}^{T-1} D_t p_t \left[(S_t' - S_t) - (R_t' - R_t) \right] - \sum_{t=1}^{T} D_t p_t \left(S_t' - S_t \right) \\
\leq \; & D_T V_T.
\end{aligned}$$

Letting $T \to \infty$ (along some subsequence of natural numbers), we obtain $C_0' \leq C_0$. The proof is completed.

Proof of Theorem 2. Suppose $G(x) = \eta x$, where $\eta > 0$ is some constant. Then $\eta^{-(t+1)} S_{t+1} = \eta^{-t}(S_t - R_t)$, $\forall t \in \mathbb{N}$. Therefore,

$$\sum_{t=0}^{\infty} \eta^{-t} R_t \leq S_0,$$

and the sequence $(\eta^{-t} S_t)_{t \in \mathbb{N}}$ is strictly decreasing and hence converges to some nonnegative number θ.

It must hold that $\theta = 0$. Otherwise, the extra amount of resource can be used in any period of time and produce more consumption goods, which can be distributed to the people in that period, and all the other periods are not affected. Then the aggregate consumption in that period is increased strictly, and the aggregate consumptions at any other times are not changed. This contradicts the dynamic efficiency of the equilibrium allocation. Thus, $\theta = 0$.

The generalized Hotelling rule implies that $D_t p_t = \eta^{-t} p_0$, $\forall t \in \mathbb{N}$. Therefore,

$$\lim_{t\to\infty} D_t p_t S_t = 0.$$

It is left to show $\lim_{t\to\infty} D_t K_{t+1} = 0$. We know that for any $t \in \mathbb{N}$,

$$C_t + K_{t+1} = F^t(K_t, N_t, R_t) = (1 + r_t)K_t + \omega_t N_t + p_t R_t,$$

where $C_t = N_t a_t + N_{t-1} b_t$. Then,

$$D_t C_t + D_t K_{t+1} = D_{t-1} K_t + D_t \omega_t N_t + D_t p_t R_t.$$

Therefore,

$$\sum_{s=0}^{t} D_s C_s + D_t K_{t+1} = D_{-1} K_0 + \sum_{s=0}^{t} D_s \omega_s N_s + p_0 \left(\sum_{s=0}^{t} \eta^{-s} R_s \right). \tag{29}$$

Since

$$\sum_{s=0}^{\infty} \eta^{-s} R_s = S_0,$$

and noticing assumption **A2**, we know that

$$\sum_{s=0}^{\infty} D_s \omega_s N_s < \infty,$$

and hence, by (29),

$$\sum_{s=0}^{\infty} D_s C_s < \infty.$$

Then, once again, by (29), we get that $\lim_{t\to\infty} D_t K_{t+1}$ exists. We prove that this limit is 0. Suppose not. Then there exist $\varepsilon > 0$ and $T \in \mathbb{N}$ such that for any $t \geq T$,

$$\sum_{s=t}^{\infty} D_s C_s < \varepsilon < D_t K_{t+1}.$$

For any $t \geq T$, let

$$\lambda_t = \frac{1}{\varepsilon} \sum_{s=t}^{\infty} D_s C_s \in (0, 1).$$

Then, for any $t \geq T$, we have

$$(\lambda_t - \lambda_{t+1}) D_t K_{t+1} > D_t C_t.$$

Now, for any $t \geq T$, let

$$K'_t = \lambda_t K_t, \quad R'_t = \lambda_t R_t, \quad C'_t = F^t(K'_t, N_t, R'_t) - K'_{t+1}.$$

We have that for any $t \geq T$,

$$\begin{aligned} D_t C'_t &= D_t F^t(K'_t, N_t, R'_t) - D_t K'_{t+1} \\ &\geq \lambda_t D_t F^t(K_t, N_t, R_t) - \lambda_{t+1} D_t K_{t+1} \\ &\geq (\lambda_t - \lambda_{t+1}) D_t K_{t+1} > D_t C_t. \end{aligned}$$

Thus, $C'_t > C_t, \quad \forall t \geq T$.

Now construct a program $(C'_t, K'_t, S'_t, R'_t)_{t\in\mathbb{N}}$ as follows: for any $t < T$, let

$$(C'_t, K'_t, S'_t, R'_t) = (C_t, K_t, S_t, R_t);$$

and for any $t \geq T$, let (C'_t, K'_t, R'_t) be constructed as above, and $(S'_t)_{T\leq t\in\mathbb{N}}$ can be constructed recursively from $(R'_t)_{t\in\mathbb{N}}$ according to the recursive equation $S'_{t+1} = \eta(S'_t - R'_t)$, $\forall t \in \mathbb{N}$.

We see that $(C_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ is dynamically improved by $(C'_t, K'_t, S'_t, R'_t)_{t\in\mathbb{N}}$. This contradicts the assumption that $(C_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ is dynamically efficient. And hence, $\lim_{t\to\infty} D_t K_{t+1} = 0$. The proof is completed.

Proof of Theorem 3. Suppose the equilibrium allocation $\mathbb{A} = (a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ is not Pareto-efficient. Then there is another allocation $\mathbb{A}' = (a'_t, b'_t, K'_t, S'_t, R'_t)_{t\in\mathbb{N}}$, which is a Pareto improvement of $\mathbb{A}$. Then we have

$$U_t(\mathbb{A}') \geq U_t(\mathbb{A}), \quad \forall t \in \mathbb{N}_-,$$

with at least one inequality being strict.

For the ancestor, we have $u(b'_0) \geq u(b_0)$, which implies $b'_0 \geq b_0$. Since $N_{-1} b_0 = (1 + r_0) K_0 + p_0 S_0$, then we have

$$N_{-1} b'_0 \geq (1 + r_0) K_0 + p_0 S_0. \tag{30}$$

Now take $t \in \mathbb{N}$ arbitrarily. For an individual of generation-t, by the definition of equilibrium, we have

$$(a_t, b_{t+1}, (S_t - R_t)) \in \arg\max_{(a,b,X)} U(a, b),$$

subject to

$$a + \frac{b}{1+r_{t+1}} \le \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}G(X)}{1+r_{t+1}} - p_t X\right),$$

then

$$(a_t, b_{t+1}) \in \arg\max_{(a,b)} U(a,b),$$

subject to

$$a + \frac{b}{1+r_{t+1}} \le \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}G(S_t - R_t)}{1+r_{t+1}} - p_t(S_t - R_t)\right).$$

Since

$$U(a'_t, b'_{t+1}) \ge U(a_t, b_{t+1}),$$

then

$$a'_t + \frac{b'_{t+1}}{1+r_{t+1}} \ge \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}G(S_t - R_t)}{1+r_{t+1}} - p_t(S_t - R_t)\right).$$

In addition, since G is concave and holds the generalized Hotelling rule:

$$\frac{p_{t+1}G'(S_t - R_t)}{1+r_{t+1}} = p_t,$$

then the function $\frac{p_{t+1}G(X)}{1+r_{t+1}} - p_t X$ of X, for given p_t, p_{t+1}, r_{t+1}, takes its maximum at $X = (S_t - R_t)$.

Therefore,

$$\frac{p_{t+1}G(S_t - R_t)}{1+r_{t+1}} - p_t(S_t - R_t) \ge \frac{p_{t+1}G(S'_t - R'_t)}{1+r_{t+1}} - p_t(S'_t - R'_t).$$

Noticing $S'_{t+1} = G(S'_t - R'_t)$, we get

$$a'_t + \frac{b'_{t+1}}{1+r_{t+1}} \ge \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}S'_{t+1}}{1+r_{t+1}} - p_t(S'_t - R'_t)\right), \quad \forall t \in \mathbb{N},$$

and hence,

$$\begin{aligned} & D_t N_t a'_t + D_{t+1} N_t b'_{t+1} \\ \ge \quad & D_t \omega_t N_t + \left(D_{t+1} p_{t+1} S'_{t+1} - D_t p_t S'_t\right) + D_t p_t R'_t, \quad \forall t \in \mathbb{N}. \end{aligned} \tag{31}$$

Now take a strict inequality from either the inequality in (30) or the inequalities in (31). There is some $\epsilon > 0$ such that adding this ϵ to the right-hand side of this inequality still preserves its validity. Then, we modify this strict inequality by adding this ϵ to its right-hand side.

Then, for sufficiently large τ, summing the inequality (30) and the inequalities in (31) for $t = 0$ through $t = \tau - 1$, yields

$$\begin{aligned} & \sum_{t=0}^{\tau-1} D_t(N_t a'_t + N_{t-1} b'_t) + D_\tau N_{\tau-1} b'_\tau \\ \geq \quad & (1 + r_0)K_0 + \sum_{t=0}^{\tau-1} D_t(\omega_t N_t + p_t R'_t) + D_\tau p_\tau S'_\tau + \epsilon. \end{aligned} \tag{32}$$

Noticing the zero maximum profit for any firm and the conditions of feasibility, we have that for any $t \in \mathbb{N}$,

$$(1 + r_t)K'_t + \omega_t N_t + p_t R'_t \geq F^t(K'_t, N_t, R'_t) \geq N_t a'_t + N_{t-1} b'_t + K'_{t+1}.$$

Therefore, for any $t \in \mathbb{N}$,

$$\begin{aligned} & D_{t-1}K'_t + D_t \omega_t N_t + D_t p_t R'_t \\ \geq \quad & D_t N_t a'_t + D_t N_{t-1} b'_t + D_t K'_{t+1}. \end{aligned} \tag{33}$$

Summing the inequalities in (33) for $t = 0$ through $t = \tau$, yields

$$\begin{aligned} & (1 + r_0)K_0 + \sum_{t=0}^{\tau} D_t \left(\omega_t N_t + p_t R'_t\right) \\ \geq \quad & \sum_{t=0}^{\tau} D_t \left(N_t a'_t + N_{t-1} b'_t\right) + D_\tau K'_{\tau+1}. \end{aligned} \tag{34}$$

By summing (32) and (34), we obtain

$$D_\tau \omega_\tau N_\tau \geq D_\tau(N_\tau a'_\tau + K'_{\tau+1}) + D_\tau p_\tau (S'_\tau - R'_\tau) + \epsilon \geq \epsilon.$$

Then we have

$$\varliminf_{\tau \to \infty} D_\tau \omega_\tau N_\tau > 0.$$

We get a contradiction. Therefore the equilibrium is Pareto-efficient. The proof is completed.

Proof of Theorem 4[22]. Suppose (4) holds. We construct a Pareto improvement of the equilibrium allocation. To this end, notice that for any $t \in \mathbb{N}$,

$$C_t + K_{t+1} = F(K_t, B_t N_t, R_t),$$

where $C_t = N_t a_t + N_{t-1} b_t$ is the total consumption at time t. Let

$$c_t = \frac{C_t}{B_t N_t}, \quad k_t = \frac{K_t}{B_t N_t}, \quad z_t = \frac{R_t}{B_t N_t}.$$

Then,

$$c_t + \mu k_{t+1} = f(k_t, z_t).$$

In the sequel, fix $(z_t)_{t\in\mathbb{N}}$. Let $x_t = \mu k_t$, $\phi(x, z) = f(x/\mu, z)$. Then,

$$c_t + x_{t+1} = \phi(x_t, z_t), \quad \forall t \in \mathbb{N}.$$

By condition (4), there exist $\varepsilon \in (0, 1)$ and $\tau \in \mathbb{N}$ such that for any $t \geq \tau$,

$$\frac{x_t \phi_x(x_t, z_t)}{x_{t+1}} < \varepsilon.$$

If we can construct a sequence $(c'_t, x'_t)_{t\in\mathbb{N}}$ such that $c'_\tau > c_\tau$, $c'_t = c_t$, $\forall t \neq \tau$, and $x'_t = x_t$, $\forall t \leq \tau$, and $x'_t > 0$, $\forall t > \tau$, then, we can get a Pareto improvement of the equilibrium allocation.

Now, fix $(x_t)_{t\leq\tau}$ and $(c_t)_{t\neq\tau}$. Let $x_{\tau+1}$ decrease a bit, then, accordingly, c_τ will increase strictly, and then, for any $t > \tau$, x_{t+1} will decrease as well.

We attempt to prove that there exists a $x'_{\tau+1} \in (0, x_{\tau+1})$ such that when $x_{\tau+1}$ decreases to $x'_{\tau+1}$, then, accordingly, for any $t > \tau$, x_{t+1} will decreases to some $x'_{t+1} > 0$.

In fact, first of all, by $\sup_{t\in\mathbb{N}} z_t < \infty$, we know that there exists $Z > 0$ such that $z_t \in [0, Z]$ for any $t \in \mathbb{N}$.

[22] The core idea lies in constructing a Pareto improvement of the equilibrium allocation. This approach was first introduced in AMZS (1989) and later reiterated in Miao (2020). However, the rigorous proof is presented here for the first time, inspired by insightful comments from Professor Zhixiang Zhang at China Economics and Management Academy, Central University of Finance and Economics, Beijing 100081, China, (email: zhangzhixiang@cufe.edu.cn), to whom the author expresses deep gratitude.

In addition, $\lim_{x\to\infty} \phi_x(x,z) < 1$ uniformly for $z \in [0,Z]$ and either $\liminf_{t\to\infty} x_t > 0$ or $\phi_x(0,z) < \infty$ for any $z \geq 0$. It follows that there exist $0 \leq \underline{x} < \overline{x} < \infty$ such that $(x_t)_{t\in\mathbb{N}}$ is bounded in $[\underline{x},\overline{x}]$ and $\phi_x(x,z)$ is well defined in $[\underline{x},\overline{x}] \times [0,Z]$; and $\min_{t\in\mathbb{N}} \phi_x(x_t,z_t) \geq \min_{t\in\mathbb{N}} \phi_x(\overline{x},z_t) =: m > 0$.

Take $\varepsilon' \in \big(0, m(\varepsilon^{-1}-1)\big)$. Noticing the uniform continuity of $\phi_x(x,z)$ in $[\underline{x},\overline{x}] \times [0,Z]$, we have that there exists $\delta \in (0,\overline{x})$ such that for any $x, x' \in [\underline{x},\overline{x}]$,

$$|\phi_x(x',z_t) - \phi_x(x,z_t)| \leq \varepsilon', \quad \forall t \in \mathbb{N},$$

if only $|x'-x| \leq \delta$.

Now, take $x'_{\tau+1}$ such that

$$0 < \frac{x_{\tau+1} - x'_{\tau+1}}{x_{\tau+1}} < \frac{\delta}{\overline{x}}.$$

Then, for any $t > \tau$,

$$\begin{aligned}
0 \quad < \quad & \frac{x_{t+1} - x'_{t+1}}{x_{t+1}} = \frac{\phi(x_t,z_t) - \phi(x'_t,z_t)}{x_{t+1}} \leq \frac{\phi_x(x'_t,z_t)(x_t - x'_t)}{x_{t+1}} \\
= \quad & \left[\frac{\phi_x(x'_t,z_t) - \phi_x(x_t,z_t)}{\phi_x(x_t,z_t)} + 1\right] \cdot \frac{x_t\phi_x(x_t,z_t)}{x_{t+1}} \cdot \frac{(x_t - x'_t)}{x_t} \\
\leq \quad & \left[\frac{\phi_x(x'_t,z_t) - \phi_x(x_t,z_t)}{\phi_x(\overline{x},z_t)} + 1\right] \cdot \frac{x_t\phi_x(x_t,z_t)}{x_{t+1}} \cdot \frac{(x_t - x'_t)}{x_t} \\
\leq \quad & \left[\frac{\phi_x(x'_t,z_t) - \phi_x(x_t,z_t)}{m} + 1\right] \cdot \frac{x_t\phi_x(x_t,z_t)}{x_{t+1}} \cdot \frac{(x_t - x'_t)}{x_t} \\
\leq \quad & \frac{x_t - x'_t}{x_t},
\end{aligned}$$

if only

$$0 < \frac{x_t - x'_t}{x_t} < \frac{\delta}{\overline{x}}.$$

It follows that $x'_t > 0$, $\forall t > \tau$. Therefore, such a construction of Pareto improvement of the equilibrium allocation is feasible. The proof is completed.

Proof of Theorem 5. For any $t \in \mathbb{N}$, by solving the utility maximization problem for an individual of generation-t, we obtain

$$N_t a_t = \frac{1}{1+\rho} I_t, \quad N_t \frac{b_{t+1}}{1+r_{t+1}} = \frac{\rho}{1+\rho} I_t.$$

We know that there exist $\varepsilon \in (0,1)$ and $T \in \mathbb{N}$ such that for any $t \geq T$,

$$\frac{1+r_t}{1+i_t} < \varepsilon.$$

Now, for any $t \geq T$, consider the function of $\theta \in [0, 1]$:

$$f_t(\theta) = \ln(a_t(1 - \theta)) + \rho \ln\left(b_{t+1} + \frac{N_{t+1}}{N_t} a_{t+1}\theta\right).$$

It's easy to see that $f_t'(\theta) > 0$ for any $\theta \in [0, \theta_t^*)$, where

$$\theta_t^* = \frac{\rho}{1+\rho}\left[1 - \frac{1+r_t}{1+i_t}\right].$$

Let

$$\theta^* = \frac{\rho}{1+\rho}(1 - \varepsilon).$$

Clearly, for any $t \geq T$,

$$\theta_t^* > \theta^*.$$

And hence for any $t \geq T$,

$$f_t(\theta^*) > f_t(0).$$

Then we can construct an allocation $(a_t', b_t', K_t, S_t, R_t)_{t\in\mathbb{N}}$, a Pareto improvement of the equilibrium allocation $(a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$, as follows: for any $t < T$,

$$a_t' = a_t, \quad b_t' = b_t;$$

and for any $t \geq T$,

$$a_t' = a_t(1 - \theta^*), \quad b_t' = b_t + \frac{N_t}{N_{t-1}} a_t \theta^*.$$

The proof is completed.

Proof of Corollary 1. Notice that $V_t \geq p_t R_t$ and Pareto efficiency is stronger than dynamic efficiency. Then, by Theorem 2 and Theorem 3, we get the required result. The proof is completed.

Proof of Corollary 2. The generalized Hotelling rule implies that for any $t \in \mathbb{N}$,

$$D_{t+1} p_{t+1} R_{t+1} = D_t p_t R_t \frac{R_{t+1}}{R_t G'(S_t - R_t)}.$$

Then, by (8), we get $\lim_{t\to\infty} D_t p_t R_t = 0$, which, by (9), yields $\lim_{t\to\infty} D_t \omega_t N_t = 0$. Thus, by Theorem 3, we obtain the required result. The proof is completed.

Proof of Corollary 3. Noticing that for any $t \in \mathbb{N}$,

$$D_t \omega_t N_t = K_0 \frac{\omega_t N_t}{(1+r_t)K_t} \prod_{s=0}^{t-1} \frac{1+j_s}{1+r_s},$$

by (10) and (11), we get $\underline{\lim}_{t\to\infty} D_t \omega_t N_t = 0$. Then, by Theorem 3, we obtain the required result. The proof is completed.

Proof of Corollary 4. The case $q < 1$ follows from Corollary 3; the case $q > 1$ follows from Theorem 4. The proof is completed.

Proof of Proposition 1. The required result follows easily from the following Lemma 5.

Proof of proposition 2. Since for any $t \in \mathbb{N}$, $K_{t+1} = \alpha\delta Y_t$, then

$$D_{t+1}Y_{t+1} = \frac{1}{\alpha} D_{t+1}(1+r_{t+1})K_{t+1} = \frac{1}{\alpha} D_t K_{t+1} = \delta D_t Y_t,$$

therefore $\lim_{t\to\infty} D_t Y_t = 0$, which yields $\lim_{t\to\infty} D_t \omega_t N_t = \lim_{t\to\infty} \beta D_t Y_t = 0$. By Theorem 3, we obtain the required result. The proof is completed.

Proof of Proposition 3. The social planner's problem ($\mathbb{P}$) is

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \delta^t \left(\delta \ln a_t + \rho \ln b_t\right), \\ \text{s.t.} \quad & K_{t+1} = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma} - N_t a_t - N_{t-1} b_t, \quad \forall t \in \mathbb{N}, \\ & S_{t+1} = \eta(S_t - R_t), \quad \forall t \in \mathbb{N}, \end{aligned}$$

and all variables are nonnegative, where K_0, S_0 are given. By transformation

$$\begin{gathered} X_t = \xi^{-t} K_t, \quad H_t = \xi^{-1/\gamma} R_t, \quad Z_t = \xi^{-1/\gamma} S_t, \\ \xi^{-(t+1)} N_t a_t = \frac{\delta}{\delta+\rho} c_t, \quad \xi^{-(t+1)} N_{t-1} b_t = \frac{\rho}{\delta+\rho} c_t, \end{gathered}$$

where $\xi = \left((1+g)(1+n)^{\beta}\right)^{1/(1-\alpha)}$, ($\mathbb{P}$) can be reduced to ($\mathbb{P}'$):

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \delta^t \ln c_t, \\ \text{s.t.} \quad & X_{t+1} = X_t^{\alpha} H_t^{\gamma} - c_t, \quad \forall t \in \mathbb{N}, \\ & Z_{t+1} = \eta(Z_t - H_t), \quad \forall t \in \mathbb{N}, \end{aligned}$$

and all variables are nonnegative, where X_0, Z_0 are given.

The Bellman equation for $(\mathbb{P}')$ is

$$V(X,Z)=\max_{c,H}\left\{\ln c+\delta V(X^\alpha H^\gamma-c,\eta(Z-H))\right\}.$$

One can verify directly that

$$V(X,Z)=\frac{1}{1-\alpha\delta}\left[\alpha\ln X+\frac{\gamma}{\tau}\ln Z\right]+m, \tag{35}$$

with some constant m, satisfies the above Bellman equation, and correspondingly, the unique solution for the optimization problem in the right-hand side of the Bellman equation is

$$c=(1-\alpha\delta)\tau^\gamma X^\alpha Z^\gamma,\quad H=\tau Z, \tag{36}$$

which is a stationary Markovian strategy for $(\mathbb{P}')$.

Denote the path of state variables by this strategy as $(X_t,Z_t)_{t\in\mathbb{N}}$, which obviously satisfies the TVCs (transversality conditions):

$$\lim_{t\to\infty}\delta^t V(X_t,Z_t)=\lim_{t\to\infty}\delta^t\left[X_tV_1(X_t,Z_t)+Z_tV_2(X_t,Z_t)\right]=0.$$

Thus, the above V in (35) is the value function of $(\mathbb{P}')$, and the strategy in (36) is the unique optimal Markovian strategy for $(\mathbb{P}')$.

Consequently, the unique optimal Markovian strategy for $(\mathbb{P})$ is as follows: for any $t\in\mathbb{N}$,

$$a_t=\frac{\beta}{1+\rho}Y_t/N_t,\quad b_t=\left(\alpha+\frac{\gamma}{\tau}\right)Y_t/N_{t-1},\quad R_t=\tau S_t,$$

where $Y_t=A_tK_t^\alpha N_t^\beta R_t^\gamma$, and the corresponding dynamics of the state variables are that for any $t\in\mathbb{N}$,

$$K_{t+1}=\alpha\delta Y_t,\quad S_{t+1}=(\eta\delta)S_t.$$

We see that the trajectory $(a_t,b_t,K_t,S_t,R_t)_{t\in\mathbb{N}}$ induced by the strategy in (36) is just the equilibrium allocation. The proof is completed.

Proof of Proposition 5. Denote the steady state of the dynamical system $\mathscr{D}$ as (k,s,z), and denote the limiting wage and the limiting interest rate of the corresponding equilibrium as ω and r, respectively. We have

$$k=\frac{1}{1+n}\left(\alpha k^\sigma+\beta+\gamma z^\sigma\right)^{(1-\sigma)/\sigma}\left[\theta-\frac{\gamma(s-z)}{z^{1-\sigma}}\right],$$

$$\alpha k^{\sigma} + \beta + \gamma z^{\sigma} = \frac{1}{1+n}\left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)}\left[\theta - \frac{\gamma(s-z)}{z^{1-\sigma}}\right].$$

It follows that

$$\omega = \beta\left(\alpha k^{\sigma} + \beta + \gamma z^{\sigma}\right)^{(1-\sigma)/\sigma} > 0,$$

$$1 + r = \alpha k^{\sigma-1}\left(\alpha k^{\sigma} + \beta + \gamma z^{\sigma}\right)^{(1-\sigma)/\sigma} = \eta > 1 + n,$$

which implies $\lim_{t\to\infty} D_t\omega_t N_t = 0$. Then, by Theorem 3, this equilibrium is Pareto-efficient. The proof is completed.

Proof of Proposition 6. First of all, no type III equilibrium exists. In fact, otherwise, by (15), for large t, approximately, $z_{t+1} = m z_t^{1-\sigma}$ with some constant $m > 0$, which yields z_t must be bounded from above. This is a contradiction.

If $\eta > 1 + n$, there dos not exist a type I equilibrium with harvesting speed indictor below θ. In fact, otherwise, for large t, approximately, $z_{t+1} = (\eta/(1+n))^{1/(1-\sigma)} z_t$. Then $z_t \not\to 0$ as $t \to \infty$. We get a contradiction. On the other hand, it's easy to verify that there exists really a unique type I equilibrium, and its harvesting speed indictor is θ, and correspondingly, its limiting capital per capita is $\pi^{-1}(\theta)$.

Now, we suppose $\eta \le 1 + n$. Under this assumption, any type I equilibrium corresponds to a $\epsilon \in [0, \theta]$ such that

$$\lim_{t\to\infty} s_t z_t^{\sigma-1} = (\theta - \epsilon)/\gamma, \quad \lim_{t\to\infty} k_t = k,$$

where $k = \pi^{-1}(\epsilon)$. In addition, for large t, approximately, by (15) and (17),

$$z_{t+1} = \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{k}{(\alpha k^{\sigma} + \beta)^{1/\sigma}} z_t.$$

Therefore,

$$\left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{k}{(\alpha k^{\sigma} + \beta)^{1/\sigma}} \le 1,$$

or, equivalently,

$$k^{\sigma}\left(\eta^{\sigma/(1-\sigma)} - \alpha^{1/(1-\sigma)}\right) \le \beta\alpha^{\sigma/(1-\sigma)}.$$

Otherwise, $z_t \not\to 0$ as $t \to \infty$. This is a contradiction. Therefore, we define

$$\overline{\theta} = \sup\left\{\epsilon \in [0, \theta] \,\middle|\, (\pi^{-1}(\epsilon))^{\sigma}\left(\eta^{\sigma/(1-\sigma)} - \alpha^{1/(1-\sigma)}\right) \le \beta\alpha^{\sigma/(1-\sigma)}\right\},$$

and define $\overline{k} = \pi^{-1}(\overline{\theta})$.

Thus, any type I equilibrium corresponds to a $k \in [0, \overline{k}]$ and a $\epsilon \in [0, \overline{\theta}]$ with $\epsilon = \pi(k)$ such that along this equilibrium path, the limit capital per capita is k, and the harvesting speed indicator is ϵ. The proof is completed.

Proof of Proposition 7. Suppose $\eta > 1 + n$. If (19) holds, then by Proposition 5, the unique type II equilibrium is Pareto-efficient. If (19) does not hold, then $\nu \leq \frac{1+n}{\eta} < 1$, which implies $\alpha \geq \eta^{\sigma} \left(2 + \rho^{-1}\right)^{\sigma-1}$. Thus, for the unique type I equilibrium, the limiting capital per capita is $k = \pi^{-1}(\theta)$, the limiting wage exists and is positive, while the limiting interest rate r satisfies

$$\frac{1+r}{1+n} = \frac{\alpha}{\theta} k^{\sigma} = x,$$

and x satisfies

$$(1+n)^{\sigma} x = \alpha \left(x + \frac{\beta}{\theta}\right)^{1-\sigma}.$$

From $\eta > 1 + n$, $\alpha \geq \eta^{\sigma} \left(2 + \rho^{-1}\right)^{\sigma-1}$, we obtain

$$(1+n)^{\sigma} < \alpha \left(1 + \frac{\beta}{\theta}\right)^{1-\sigma}.$$

Therefore, $x > 1$. Then, by Corollary 4, this equilibrium is Pareto-efficient.

Suppose $\eta \leq 1 + n$. Taking $k \in (0, \overline{k}]$ and $\epsilon \in (0, \overline{\theta}]$ with $\epsilon = \pi(k)$ arbitrarily. The type I equilibrium with limiting capital per capita k satisfies the assumption **A3**. Denote its limiting interest rate as r, and let

$$\phi = \frac{1+r}{1+n}.$$

From

$$1 + r = \alpha k^{\sigma-1} \left(\alpha k^{\sigma} + \beta\right)^{(1-\sigma)/\sigma},$$

we obtain

$$(1+n)^{\sigma} \phi = \alpha \left(\phi + \frac{\beta}{\epsilon}\right)^{1-\sigma}.$$

It's easy to see that there exists a $\underline{\theta} \in (0, \overline{\theta}]$ such that $\phi > 1$, if $\epsilon < \underline{\theta}$; $\phi < 1$, if $\epsilon > \underline{\theta}$. Denote $\underline{k} = \pi^{-1}(\underline{\theta})$. Then, by Corollary 4, this equilibrium is Pareto-efficient, if $k < \underline{k}$; it is Pareto-inefficient, if $k > \underline{k}$. The proof is completed.

Proof of Proposition 8. (i) Suppose $\eta \leq 1+n$. It's easy to see that there is no equilibrium of type III, but there is a continuum of equilibria of type I: for any $\epsilon \in [0, \theta]$, there is an equilibrium of type I such that as $t \to \infty$,

$$\frac{s_t - z_t}{z_t^{1-\sigma}} \to (\theta - \epsilon)/\gamma, \quad k_t \to 0.$$

(ii) Suppose $\eta > 1+n$. First of all, there is no type III equilibrium with harvesting speed indictor 0. In fact, otherwise, by (16), we have that for large t, $z_{t+1} \leq z_t/2$, which contradicts $z_t \to \infty$ as $t \to \infty$.

If there is a type III equilibrium with harvesting speed indictor $\epsilon \in (0, \theta]$ and the limiting capital per capita $k > 0$, then, as $t \to \infty$,

$$\frac{s_t - z_t}{z_t^{1-\sigma}} \to \frac{\theta - \epsilon}{\gamma}, \quad k_t \to k,$$

and

$$k = \frac{\epsilon}{1+n} \left(\alpha k^\sigma + \beta\right)^{(1-\sigma)/\sigma}.$$

For simplicity, let $x = \frac{\alpha}{\epsilon} k^\sigma$. Then,

$$(1+n)^\sigma x = \alpha \left(x + \frac{\beta}{\epsilon}\right)^{1-\sigma}. \tag{37}$$

In addition, by (17), for large t, approximately,

$$z_{t+1} = \frac{\epsilon}{1+n} \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{z_t}{\alpha k^\sigma + \beta},$$

then it must hold that

$$1 \leq \frac{\epsilon}{1+n} \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{1}{\alpha k^\sigma + \beta} = \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{1}{(1+n)\left[x + \frac{\beta}{\epsilon}\right]},$$

or, equivalently,

$$\frac{\eta}{\alpha} \geq (1+n)^{1-\sigma} \left[x + \frac{\beta}{\epsilon}\right]^{1-\sigma} = (1+n)\frac{x}{\alpha},$$

that is,

$$x \leq \frac{\eta}{1+n}. \tag{38}$$

To sum up, there is a type III equilibrium with harvesting speed indicator $\epsilon \in (0, \theta]$, if and only if (37) has a solution satisfying (38).

For given ϵ, the equation (37) for x has a solution, if and only if $\epsilon \geq \epsilon_*$, which implies $\theta \geq \epsilon_*$, or, equivalently, $\alpha \leq \alpha^*$. In addition, since (19) is not satisfied, then,

$$\frac{\eta}{1+n}\left[\left(\frac{\eta^\sigma}{\alpha}\right)^{1/(1-\sigma)} - 1\right] \leq \frac{\beta}{\theta}.$$

With the above observations, the remainder of the results can be proven. The proof is completed.

Proof of Proposition 9. First, consider an equilibrium of type I. Since for any $t \in \mathbb{N}$,

$$1 + r_t = \alpha k_t^{\sigma-1}\left(\alpha k_t^\sigma + \beta + \gamma z_t^\sigma\right)^{(1-\sigma)/\sigma},$$

$$\omega_t = \beta\left(\alpha k_t^\sigma + \beta + \gamma z_t^\sigma\right)^{(1-\sigma)/\sigma},$$

then,

$$\begin{aligned} \frac{1}{1+r_{t+1}}\frac{\omega_{t+1}N_{t+1}}{\omega_t N_t} &= \frac{1+n}{\alpha}\frac{k_{t+1}^{1-\sigma}}{\left(\alpha k_t^\sigma + \beta + \gamma z_t^\sigma\right)^{(1-\sigma)/\sigma}} \\ = \quad & \frac{k_{t+1}^{-\sigma}}{\alpha}\left[\theta - \frac{\gamma(s_t - z_t)}{z_t^{1-\sigma}}\right] \leq \frac{\theta}{\alpha}k_{t+1}^{-\sigma} \to 0, \quad \text{as} \quad t \to \infty, \end{aligned}$$

which implies $\lim_{t\to\infty} D_t\omega_t N_t = 0$. Then, by Theorem 3, the equilibrium is Pareto-efficient.

Next, for any type III equilibrium with the harvesting speed indicator $\epsilon \in [\epsilon^*, \theta]$ and limiting capital per capita k and limiting interest rate r, we have

$$\frac{1+r}{1+n} = x = \frac{\alpha}{\epsilon}k^\sigma,$$

where x satisfies

$$(1+n)^\sigma x = \alpha\left(x + \frac{\beta}{\epsilon}\right)^{1-\sigma}.$$

According to Corollary 4, this equilibrium is Parto-efficient if $x > 1$ and Pareto-inefficient if $x < 1$. Therefore, by determining whether $x > 1$ or $x < 1$, one can prove the remainder of the result.

Proof of Proposition 11. It's easy to verify that if $\beta < \kappa$, then, $x^* \in \left(0, \frac{B}{2}\left(1 - \frac{1}{\lambda}\right)\right)$, which implies $G'(x^*) > 1$; if $\beta > \kappa$, then $x^* \in \left(\frac{B}{2}\left(1-\frac{1}{\lambda}\right), \frac{B}{2}\right)$, which implies $G'(x^*) < 1$.

For any $t \in \mathbb{N}$, denote $x_t = S_t - R_t$. Noticing that as $t \to \infty$,

$$x_t \to x^*, \quad R_t \to R^*, \quad S_t \to S^*,$$

and for any $t \in \mathbb{N}$,

$$\begin{aligned} I_t &= \omega_t N_t + \left[\frac{p_{t+1} G(x_t)}{1 + r_{t+1}} - p_t x_t \right] \\ &= \omega_t N_t + p_t \left[\frac{G(x_t)}{G'(x_t)} - x_t \right] \\ &= Y_t \left\{ \beta + \frac{\gamma}{R_t} \left[\frac{G(x_t)}{G'(x_t)} - x_t \right] \right\}, \end{aligned}$$

by (23) and (25), we have

$$\begin{aligned} & \lim_{t\to\infty} \frac{1 + i_t}{1 + r_t} = \lim_{t\to\infty} \frac{I_t}{(1 + r_t) I_{t-1}} = \lim_{t\to\infty} \frac{Y_t}{(1 + r_t) Y_{t-1}} \\ = & \lim_{t\to\infty} \frac{K_{t+1}}{(1 + r_t) K_t} = \lim_{t\to\infty} \frac{K_{t+1}}{\alpha Y_t} = \lim_{t\to\infty} \frac{R_{t+1}}{R_t G'(x_t)} = \frac{1}{G'(x^*)}. \end{aligned}$$

If $\beta > \kappa$, then

$$\lim_{t\to\infty} \frac{1 + i_t}{1 + r_t} > 1,$$

thus, by Theorem 5, the equilibrium is Pareto-inefficient.

If $\beta < \kappa$, then

$$\lim_{t\to\infty} \frac{Y_t}{(1 + r_t) Y_{t-1}} < 1,$$

therefore,

$$\lim_{t\to\infty} D_t \omega_t N_t = \beta \lim_{t\to\infty} D_t Y_t = 0,$$

thus, by Theorem 3, the equilibrium is Pareto-efficient. The proof is completed.