On the Optimal Design of All-Pay Auctions*

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Abstract

We consider the optimal design of complete-information all-pay auctions with multiple heterogeneous players when a designer can manipulate contestants' relative competitiveness by imposing identity-dependent treatments. Two types of instruments are considered: (i) multiplicative biases that assign individualized weights to each contender's effective effort entry and (ii) additive headstarts that directly add to it. We show that, in general, both instruments will be used in the optimum. Moreover, the contest designer is able to induce every allocation of the prize while achieving full surplus extraction with an appropriately designed contest rule and tie-breaking rule.

Keywords: All-pay Auction; Biases; Headstarts; Favoritism.

JEL Classification Codes: C72, D44, D72.

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1 Introduction

A wide variety of competitive activities resemble a contest. Politicians campaign for votes; interest groups lobby for policy influence; firms race for technological breakthroughs that can be patented; and students engage in academic efforts to secure seats at premium colleges. In all of these scenarios, contenders expend costly effort to vie for limited prizes, while the competitive outlays are nonrefundable regardless of the outcome.

A voluminous body of economics literature has examined contestants' strategic behavior and the optimal design of contests. An all-pay auction—which fully rewards superior effort—is an intuitive framework for modeling the prize allocation mechanism. It awards the prize to the highest bidder with certainty: In its simplest form, a contestant wins the contest with probability one if his effort x_i exceeds those of the others, i.e.,

$$p_i(\boldsymbol{x}) = 1, \text{ if } x_i > x_j, \forall j \neq i,$$

for a given set of effort entries $\boldsymbol{x} := (x_1, \ldots, x_n)$.

In this paper, we explore the optimal design of all-pay auctions when a contest designer is able to award identity-dependent preferential treatments to contestants to manipulate the competitive balance of the playing field. The economics literature has long espoused the strategic use of preferential treatments tailored to individual characteristics to incentivize effort supply: A contest designer can strategically favor or handicap contestants to bias the competition to promote her own interests (e.g., Siegel, 2014; Szech, 2015).[1] The prevalence of this practice is evidenced by the numerous examples documented in the literature.[2]

Two instruments are broadly adopted in the literature to model the biases imposed on contestants' effort entries: (i) multiplicative biases and (ii) additive headstarts.[3] The former—e.g., Fu (2006) and Epstein, Mealem, and Nitzan (2011)—places a fixed weight on a contestant's effort, while the latter—e.g., Kirkegaard (2012) and Pastine and Pastine (2012)—directly adds to it. In a biased all-pay auction, each contestant's effort is adjusted

[1] See also Mealem and Nitzan (2016); Chowdhury, Esteve-González, and Mukherjee (2020); and Fu and Wu (2019) for thorough surveys of this strand of the literature.

[2] Consider, for instance, government policies that favor small and medium-sized enterprises (SMEs) in public procurement to support local entrepreneurship in various countries (Che and Gale, 2003; Epstein, Mealem, and Nitzan, 2011). Prestigious colleges often allocate bonus points to minority applicants when practicing affirmative action in admissions (Fu, 2006; Franke, 2012). In competitions for vacant positions, internal candidates are often ex ante preferred to external candidates to incentivize productive efforts (Chan, 1996). In a corporate succession race, the leading candidate is often awarded a key appointment—e.g., a president or chief operating officer—that allows him/her privileged access to corporate resources in carrying out assigned tasks (Fu and Wu, 2021).

[3] Other design instruments considered in the literature include bid caps and taxes/subsidies. See, e.g., Che and Gale (1998, 2006); Glazer and Konrad (1999); Gavious, Moldovanu, and Sela (2002); Kaplan and Wettstein (2006); Pastine and Pastine (2013); Mealem and Nitzan (2014); Olszewski and Siegel (2019); Fu, Wu, and Zhu (2021); and Cohen, Darioshi, and Nitzan (2021), among others.

by the biases and converted into a score, and the highest scoring contestant wins the prize.

We consider a multi-player all-pay auction and allow the designer to use both instruments to optimize toward a general design objective. Fu and Wu (2020) and Deng, Fu, and Wu (2021) develop an indirect optimization approach for the design of biased lottery contests. We adapt this approach to the setting of all-pay auctions. This allows us to (i) characterize the feasibility frontier of the contest under a general objective function, and then (ii) demonstrate that an optimally designed all-pay auction, with the use of both multiplicative biases and headstarts, can achieve the feasibility frontier and fully extract each contestant's surplus. The result also implies that a properly designed all-pay auction outperforms all possible contest mechanisms that yield pure-strategy equilibria.[4]

Our paper extends the literature in three ways. First, we construct a general objective function that encompasses a broad array of scenarios. The literature on contest design typically focuses on specific objective functions, with the majority aiming to maximize total effort. Examples include Kirkegaard (2012); Li and Yu (2012); Franke, Kanzow, Leininger, and Schwartz (2014); and Franke, Leininger, and Wasser (2018). However, the pursuit of alternative objectives is not uncommon in practice. Consider, for instance, that a college presumably only cares about the academic quality of its admitted student body (see Fu, 2006). In a crowdsourcing competition for a technical solution, the buyer would only value the quality of the winning entry. It is easier to promote a sporting event when there is more suspense about its outcome (see Chan, Courty, and Hao, 2008). Alternatively, in public procurement, a government (as a buyer) could be concerned about domestic suppliers' efforts and also (as a social planner) concerned about their welfare (see Epstein, Mealem, and Nitzan, 2011). We construct an objective function that encompasses all of these concerns.

Second, our analysis departs from the usual two-player setting and allows for an arbitrary number of contestants. Equilibrium analysis of all-pay auctions with three or more players poses a technical challenge when contestants are heterogeneous and biases can be imposed. Fu and Wu (2020) develop an alternative technique that bypasses the analytical difficulty in generalized lottery contests. In this paper, we revive the equilibrium characterization result of Baye, Kovenock, and De Vries (1996), which further allows us to adapt the approach of Fu and Wu (2020) to all-pay auctions.

Third, thanks to the above-mentioned optimization approach, our analysis allows the designer to choose an arbitrary combination of multiplicative biases and headstarts. In the majority of prior studies of optimally biased contests, the designer is endowed with a single instrument (e.g., Franke, Kanzow, Leininger, and Schwartz, 2014). We demonstrate that the optimum, in general, requires that the two instruments be imposed together. Notable

[4]Franke, Leininger, and Wasser (2018) show that an all-pay auction, with a proper combination of multiplicative biases and headstarts, can achieve the first best. However, they only consider the maximization of total effort.

exceptions include Kirkegaard (2012); Franke, Leininger, and Wasser (2018); and Zhu (2021). However, all of these studies focus on specific objective functions. Kirkegaard (2012) and Franke, Leininger, and Wasser (2018) consider total effort maximization, while Zhu (2021) also considers maximization of the maximum effort.

The rest of the paper is organized as follows. Section 2 sets up the contest model and describes the objective function for contest design. Section 3 conducts the analysis and discusses its implications, and Section 4 concludes. Proofs are collected in the Appendix.

2 The Model

There are $n \geq 2$ risk-neutral contestants competing for a prize. The prize has a value $v_i > 0$ for each contestant $i \in \mathcal{N} \equiv \{1, \ldots, n\}$—with $v_1 \geq \cdots \geq v_n$—which is commonly known. To win the prize, contestants simultaneously commit to their efforts $x_i \geq 0$. One's bid incurs a unity marginal effort cost.

Winner-selection Mechanism and Design Instruments Fixing a set of effort entries $\boldsymbol{x} \equiv (x_1, \ldots, x_n) \geq (0, \ldots, 0)$, let us denote by $p_i(x_i, \boldsymbol{x}_{-i})$ a contestant i's probability of winning the contest, where $\boldsymbol{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is the effort profile of his rivals. Contestant i's probability of winning the contest—i.e., the contest success function (CSF)—is given by

$$p_i(x_i, \boldsymbol{x}_{-i}) = \begin{cases} 1 & \text{if } \alpha_i x_i + \beta_i > \max_{j \neq i} \left\{ \alpha_j x_j + \beta_j \right\}, \\ \frac{1}{m} & \text{if } \alpha_i x_i + \beta_i \text{ is among the } m \text{ highest of } \left\{ \alpha_j x_j + \beta_j \right\}_{j=1}^{n} \text{ with a tie}, \\ 0 & \text{if } \alpha_i x_i + \beta_i < \max_{j \neq i} \left\{ \alpha_j x_j + \beta_j \right\}, \end{cases} \tag{1}$$

where $\alpha_i \geq 0$ and $\beta_i \geq 0$ are the *multiplicative biases* and *additive headstarts* the designer imposes on each contestant $i \in \mathcal{N}$, respectively. A contestant i wins the contest if his effective output or score—i.e., $\alpha_i x_i + \beta_i$—exceeds those of all the others; when multiple players have the same highest effective output or score, they win the contest with an equal probability.[5] Contestant i's expected payoff can then be written as

$$\pi_i(x_i, \boldsymbol{x}_{-i}) := p_i(x_i, \boldsymbol{x}_{-i}) \cdot v_i - x_i, \text{for all } i \in \mathcal{N}.$$

Both instruments are popularly adopted in the literature to model preferential treatments. For multiplicative biases, see Fu (2006); Franke (2012); Franke, Kanzow, Leininger, and

[5] We assume a symmetric tie-breaking rule in the CSF (1). It should be noted that the contest designer in our model is allowed to manipulate the tie-breaking rule. We will inform the readers when the designer does so. See, for example, Remark 1.

Schwartz (2014); and Epstein, Mealem, and Nitzan (2011). For headstarts, see Clark and Riis (2000); Konrad (2002); Siegel (2009, 2014); Li and Yu (2012); and Seel and Wasser (2014). Kirkegaard (2012); Franke, Leininger, and Wasser (2018); and Zhu (2021) allow for both.

Contest Objective It is well known that a complete-information all-pay auction, in general, does not have pure-strategy equilibria. Let $G_i(\hat{x}_i)$ denote an arbitrary cumulative distribution function (CDF) representing the mixed-strategy of player i; let S_i denote the support of this distribution. Fixing an arbitrary strategy profile $\langle G_1(x_1), \ldots, G_n(x_n)\rangle$, denote by x_i^e and p_i^e, respectively, contestant i's expected effort $\int_{x_i \in S_i} x_i dG_i(x_i)$ and expected winning probability $\int \cdots \int_{\boldsymbol{x} \in \times_{i=1}^n S_i} p_i(x_i, \boldsymbol{x}_{-i}) dG_1(x_1) \cdots dG_n(x_n)$.

We assume that the contest designer's objective function, which we denote by $\Lambda(\cdot)$, is a function of the profile of expected efforts $\boldsymbol{x}^e := (x_1^e, \ldots, x_n^e)$; the profile of expected winning probabilities $\boldsymbol{p}^e := (p_1^e, \ldots, p_n^e)$; and the profile of contestants' prize valuations $\boldsymbol{v} := (v_1, \ldots, v_n)$. The following assumption is imposed on the objective function $\Lambda(\cdot)$ throughout the paper.

Assumption 1 *Fixing $\boldsymbol{p}^e \equiv (p_1^e, \ldots, p_n^e)$ and $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$, $\Lambda(\boldsymbol{x}^e, \boldsymbol{p}^e, \boldsymbol{v})$ is weakly increasing in x_i^e for all $i \in \mathcal{N}$.*

Two remarks are in order. First, Assumption 1 simply requires that contestants' expected efforts accrue to the benefit of the contest designer, holding fixed contestants' expected winning probabilities and prize valuations. Note that both $\boldsymbol{x}^e \equiv (x_1^e, \ldots, x_n^e)$ and $\boldsymbol{p}^e \equiv (p_1^e, \ldots, p_n^e)$ are defined over contestants' strategy profile $\langle G_1(x_1), \ldots, G_n(x_n)\rangle$. Therefore, changing contestants' mixed-strategy profile in a way that changes $\boldsymbol{x}^e$ will also change $\boldsymbol{p}^e$ in general.

Second, Assumption 1 specifies a mild regularity condition, and the objective function $\Lambda(\cdot)$ encompasses a broad array of scenarios for contest design. It can be satisfied by many popularly studied objective functions in the literature. The following example demonstrates the versatility of $\Lambda(\cdot)$.

Example 1 *The following objective function satisfies Assumption 1:*

$$\Lambda(\boldsymbol{x}^e, \boldsymbol{p}^e, \boldsymbol{v}) := \sum_{i=1}^n x_i^e + \lambda \sum_{i=1}^n p_i^e v_i - \gamma \sum_{i=1}^n \left(p_i^e - \frac{\sum_{i=1}^n p_i^e}{n} \right)^2, \text{ with } \lambda, \gamma \geq 0. \tag{2}$$

In the case of $\lambda = \gamma = 0$, the above expression degenerates to $\Lambda(\boldsymbol{x}^e, \boldsymbol{p}^e, \boldsymbol{v}) = \sum_{i=1}^n x_i^e$, i.e., maximization of expected total effort, which is the most widely assumed objective for contest design in the literature. In addition to expected effort supply, the contest designer may be *concerned with selection efficiency and/or the closeness of the competition.*[6] *The former is captured by the term $\sum_{i=1}^{n} p_i^e v_i$, which is the expected prize valuation of the winner. The contest objective accommodates the concern about selection efficiency when $\lambda > 0$. Note that the concern for selection efficiency also alludes to a preference for contestants' welfare (see Epstein, Mealem, and Nitzan, 2011): Ceteris paribus, contestants' aggregate welfare—i.e., $\sum_{i=1}^{n} (p_i^e v_i - x_i^e)$—improves when the prize is distributed to the one with the highest valuation.*

The latter is captured by the term $\sum_{i=1}^{n} \left[p_i^e - (\sum_{i=1}^{n} p_i^e)/n\right]^2$, which depicts a typical scenario in the administration of sporting events: Spectators often not only appreciate contenders' efforts, but also demand suspense about the contest outcome. The term is the variance of the expected equilibrium winning probability profile, which measures the predictability of the competitive event. The objective function thus reflects the designer's preference for a closer race when $\gamma > 0$.

Contest Design Prior to the competition, the designer, anticipating contestants' equilibrium bidding strategies, chooses and commits to a contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \langle(\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)\rangle \geq \langle(0, \ldots, 0), (0, \ldots, 0)\rangle$ to maximize $\Lambda(\cdot)$. Therefore, the optimal contest design problem yields a constrained optimization problem. A change in the contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ would reshape contestants' equilibrium bidding strategies, which in turn would vary their expected equilibrium efforts and winning probabilities.

As noted by Franke, Leininger, and Wasser (2018), a complete characterization of the set of equilibria of all-pay auctions with headstarts has yet to be provided in the literature. Further, Baye, Kovenock, and De Vries (1993, 1996) show that there may exist a continuum of mixed-strategy equilibria in an unbiased all-pay auction with three or more players. In what follows, we assume that (i) the contest designer is restricted to choosing from the set of contest rules under which a (mixed-strategy) Nash equilibrium exists; and (ii) the equilibrium most favorable to the contest designer is selected when multiple equilibria exist.

3 Analysis and Results

The studies on the optimal design of complete-information all-pay auctions typically assume two players and employs a direct brute-force approach: They first solve for the unique equilibrium bidding strategy for any given contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, insert the solution into the objective function, then search for the optimal rule (e.g., Epstein, Mealem, and Nitzan, 2011;

[6] For contest design for selection efficiency, see Meyer (1991); Hvide and Kristiansen (2003); Ryvkin and Ortmann (2008); and Fang and Noe (2021). For economics studies of suspense in competition, see Fort and Quirk (1995); Szymanski (2003); Runkel (2006); Chan, Courty, and Hao (2008); and Ely, Frankel, and Kamenica (2015).

Li and Yu, 2012; Zhu, 2021). This approach relies on the equilibrium characterization and cannot be applied to the multi-player setting ($n \geq 3$) because, as mentioned previously, a complete equilibrium characterization of a biased multi-player all-pay auction is technically challenging and remains absent in the literature.[7]

To overcome the aforementioned difficulty, the literature usually takes an indirect constructive approach. For instance, Franke, Kanzow, Leininger, and Schwartz (2014) investigate the effort-maximizing multiplicative biases. They first establish an upper bound and a lower bound for the contest performance, then show that the two bounds coincide. Similarly, Franke, Leininger, and Wasser (2018) search for the optimal combinations of multiplicative biases and headstarts that maximize the expected total effort. Again, they construct a contest rule to achieve the maximum expected total effort (revenue), which corresponds to the highest prize valuation among the contestants. Their constructions are effective when total effort (revenue) is the focus, but may lose value when alternative objectives are pursued for contest design.

Our analysis borrows from the indirect approach proposed by Fu and Wu (2020) and Deng, Fu, and Wu (2021), which can be summarized as follows. Instead of focusing on contestants' equilibrium effort profile under a contest rule, we take a detour and focus on the expected equilibrium winning probability profile. Specifically, we show that all expected winning probability profiles—except those in which some contestant wins the contest with probability one—can be induced by some contest rule under a symmetric tie-breaking rule. If the designer is allowed to manipulate the tie-breaking rule, then all expected winning probability profiles can be induced in equilibrium. We then demonstrate that we can maintain an expected equilibrium winning probability profile, while modifying the contest rule to fully extract surplus from each contestant, which closes the loop. The detail will be revealed in Theorem 1 and its sketch proof.

In the remainder of the section, we first carry out the analysis to characterize the main result, and then we discuss the implications of our results in relation to the literature.

3.1 Optimal All-pay Auction

Before we proceed to the formal analysis, it is useful to state the following.

Definition 1 ***(Feasible Effort Profile)*** *An expected effort profile* $\boldsymbol{x}^e \equiv (x_1^e, \ldots, x_n^e)$ *is feasible for the expected winning probability profile* $\boldsymbol{p}^e \in \Delta^{n-1}$ *if there exists a contest rule* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq (\mathbf{0}, \mathbf{0})$ *and an equilibrium for this contest rule that generates the expected effort profile* $\boldsymbol{x}^e$ *and leads to* $\boldsymbol{p}^e$.

[7] See Siegel (2009, 2014) and Franke, Leininger, and Wasser (2018) for important results on equilibrium characterization.

Suppose $\boldsymbol{\beta} = \mathbf{0}$. Let $\hat{v}_i := \alpha_i v_i$ and $\hat{\boldsymbol{v}} := (\hat{v}_1, \ldots, \hat{v}_n)$. Denote by $\widehat{G}_i(x_i)$ the CDF representing the equilibrium mixed-strategy of player i and by $\widehat{S}_i$ the support of the distribution in an unbiased contest, i.e., $\alpha_i = \alpha_j > 0$ for all $i, j \in \mathcal{N}$. Further, denote by $\hat{x}_i^e$ and $\hat{p}_i^e$, respectively, contestant i's expected effort $\int_{x_i \in \widehat{S}_i} x_i d\widehat{G}_i(x_i)$ and expected winning probability $\int \cdots \int_{\boldsymbol{x} \in \times_{i=1}^n \widehat{S}_i} p_i(x_i, \boldsymbol{x}_{-i}) d\widehat{G}_1(x_1) \cdots d\widehat{G}_n(x_n)$. The following result presented by Franke, Kanzow, Leininger, and Schwartz (2014) allows us to transform the biased all-pay auction with zero headstarts—i.e., with $\alpha_i \neq \alpha_j$ for some $i, j \in \mathcal{N}$—into a standard unbiased all-pay auction.

Lemma 1 *Consider a biased all-pay auction contest with zero headstarts. For every equilibrium strategy profile $\langle G_1(x_1), \ldots, G_n(x_n)\rangle$ under $\langle \boldsymbol{v}, \boldsymbol{\alpha}\rangle$, there exists an equilibrium strategy profile $\langle \widehat{G}_1(x_1), \ldots, \widehat{G}_n(x_n)\rangle$ under $\langle \hat{\boldsymbol{v}}, \hat{\boldsymbol{\alpha}}\rangle := \langle(\alpha_1 v_1, \ldots, \alpha_n v_n), (1, \ldots, 1)\rangle$ such that $\hat{x}_i^e = \alpha_i x_i^e$ for all $i \in \mathcal{N}$. Moreover, the equilibrium strategy profile $\langle G_1(x_1), \ldots, G_n(x_n)\rangle$ under the contest rule $\boldsymbol{\alpha}$ and $\langle \widehat{G}_1(x_1), \ldots, \widehat{G}_n(x_n)\rangle$ under $\hat{\boldsymbol{\alpha}}$ lead to the same profile of expected winning probabilities, i.e., $(p_1^e, \ldots, p_n^e) = (\hat{p}_1^e, \ldots, \hat{p}_n^e)$.*

Lemma 1 unveils the strategic equivalence between the biased all-pay auction and the transformed unbiased counterpart, which revives the equilibria characterization result of Baye, Kovenock, and De Vries (1996) in our setting. We obtain the following key result.

Theorem 1 *Consider all-pay auctions with a CSF as specified in* (1) *and fix an arbitrary $\boldsymbol{p}^e \in \Delta^{n-1}$ such that $p_i^e \neq 1$ for all $i \in \mathcal{N}$. Then $\boldsymbol{x}^e \equiv (x_1^e, \ldots, x_n^e) = \boldsymbol{p}^e \circ \boldsymbol{v} = (p_1^e v_1, \ldots, p_n^e v_n)$ is feasible for $\boldsymbol{p}^e$.*

Theorem 1 states that for every expected winning probability profile $\boldsymbol{p}^e$ such that $p_i^e \neq 1$ for all $i \in \mathcal{N}$, there exists a contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ that induces $\boldsymbol{p}^e$ and a profile of expected efforts $\boldsymbol{x}^e \equiv (p_1^e v_1, \ldots, p_n^e v_n)$. Obviously, each player's participation constraint binds under this contest rule, indicating that the maximum expected effort is achieved.

The result is proven by construction. A sketch proof is laid out in the main text: It elucidates the different roles played by the multiplicative biases and headstarts in this context and also helps us understand the comparison between all-pay auctions and noisy contests, which we further elaborate on in Section 3.2. For ease of exposition, let us consider the case in which $p_i^e \neq 1$ for all $i \in \mathcal{N}$ and $p_1^e > p_2^e \geq \cdots \geq p_n^e$.[8] The sketch proof proceeds in the following two steps:

Step I (Introducing Multiplicative Biases): Fix an arbitrary expected equilibrium winning probability profile $\boldsymbol{p}^e \in \Delta^{n-1}$ with $p_i^e \neq 1$ for all $i \in \mathcal{N}$, and set headstarts to

[8]Step II is unnecessary for the proof of Theorem 1 in the case $p_1^e = p_2^e \geq \cdots \geq p_n^e$. In other words, headstarts are not used if the contest designer aims to induce a profile of expected winning probabilities in which the highest equilibrium winning probability is equal to the second highest one.

zero. We can construct a set of multiplicative biases $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$ that satisfies $\alpha_1^* v_1 > \alpha_2^* v_2 = \cdots = \alpha_n^* v_n > 0$, such that there exists a mixed-strategy equilibrium that leads to the given expected equilibrium winning probabilities $\boldsymbol{p}^e \equiv (p_1^e, \ldots, p_n^e)$. By our construction, contestants 2 to n each earn an expected payoff of zero, and player 1 receives a positive expected payoff of size $(\alpha_1^* v_1 - \alpha_2^* v_2)/\alpha_1^*$.

Step II (Introducing Additive Headstarts): We add headstarts to the contest rule to further incentivize player 1 without disturbing the equilibrium incentives of the other players. Consider the following set of contest rules $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger) := \langle(\alpha_1^\dagger, \ldots, \alpha_n^\dagger), (\beta_1^\dagger, \ldots, \beta_n^\dagger)\rangle$:

$$\left(\alpha_i^\dagger, \beta_i^\dagger\right) := \begin{cases} \left(\alpha_1^*, 0\right) & \text{for } i = 1, \\ \left(\alpha_i^*, \alpha_1^* v_1 - \alpha_2^* v_2\right) & \text{for } i \in \{2, \ldots, n\}. \end{cases}$$

In words, we give the same headstarts to all players except for player 1. Compared with the equilibrium constructed in Step I, contestant 1's equilibrium effort distribution is shifted upward by $(\alpha_1^* v_1 - \alpha_2^* v_2)/\alpha_1^*$, whereas all other players' equilibrium strategies remain unchanged. The additional effort supply from contestant 1 completely offsets the preferential treatment awarded to other players through the headstart, and the size of the headstarts is chosen to fully deplete the surplus left to contestant 1 in the equilibrium, i.e., earning zero expected payoff in the contest. This, in turn, implies that $x_i^e = p_i^e v_i$ for all $i \in \mathcal{N}$ in equilibrium under the contest rule $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger)$.

Note that Theorem 1 requires $p_i^e \neq 1$ for all $i \in \mathcal{N}$. This requirement is caused by the restriction of the symmetric tie-breaking rule in the CSF specified in (1) and can be dropped if we allow the contest designer to modify the prevailing tie-breaking rule, as in Szech (2015) and Franke, Leininger, and Wasser (2018).

Remark 1 *Consider the following modified CSF: Fixing an arbitrary effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, a player s's winning probability is given by*

$$p_s(x_s, \boldsymbol{x}_{-s}) = \begin{cases} 1 & \textit{if } \alpha_s x_s + \beta_s \geq \max_{j \neq s} \left\{\alpha_j x_j + \beta_j\right\}, \\ 0 & \textit{if } \alpha_s x_s + \beta_s < \max_{j \neq s} \left\{\alpha_j x_j + \beta_j\right\}. \end{cases}$$

The probability of player $k \neq s$ winning the contest is

$$p_k(x_k, \boldsymbol{x}_{-k}) = \begin{cases} 1 & \textit{if } \alpha_k x_k + \beta_k > \max_{j \neq k} \left\{\alpha_j x_j + \beta_j\right\}, \\ \frac{1}{m} & \textit{if } \alpha_k x_k + \beta_k \textit{ is among the } m \textit{ highest of } \left\{\alpha_j x_j + \beta_j\right\}_{j=1}^n \textit{ with a tie,} \\ & \textit{and } \alpha_k x_k + \beta_k \neq \alpha_s x_s + \beta_s, \\ 0 & \textit{if } \alpha_k x_k + \beta_k < \max_{j \neq k} \left\{\alpha_j x_j + \beta_j\right\} \textit{ or } \alpha_k x_k + \beta_k = \alpha_s x_s + \beta_s. \end{cases}$$

In words, the tie-breaking rule in the above CSF favors player s: He wins the contest with certainty whenever his effective output or score $\alpha_s x_s + \beta_s$ is the highest among all contestants. It can be verified that player s exerting effort v_s and all other players remaining inactive constitute a Nash equilibrium of the contest game under the contest rule where $(\alpha_s, \beta_s) = (1, 0)$ and $(\alpha_k, \beta_k) = (0, v_s)$ for all $k \neq s$.[9] *In the equilibrium, $p_s^e = 1$ and $p_k^e = 0$ for all $k \neq s$, from which we can conclude that the expected effort profile $\boldsymbol{x}^e = (p_1^e v_1, \ldots, p_n^e v_n)$ is feasible for $\boldsymbol{p}^e \equiv (p_1^e, \ldots, p_{s-1}^e, p_s^e, p_{s+1}^e, \ldots, p_n^e) = (0, \ldots, 0, 1, 0, \ldots, 0)$.*

Theorem 1 and Remark 1 enable us to reformulate the designer's optimization problem as follows: Under Assumption 1, $\boldsymbol{x}^e$ can be replaced by $\boldsymbol{p}^e \circ \boldsymbol{v}$ and she chooses the expected winning probability profile $\boldsymbol{p}^e \equiv (p_1^e, \ldots, p_n^e)$ as the design variable to maximize an objective function $\Lambda\big(\boldsymbol{p}^e \circ \boldsymbol{v}, \boldsymbol{p}^e, \boldsymbol{v}\big)$, subject to the constraint $\boldsymbol{p}^e \in \Delta^{n-1}$.

Next, we continue with Example 1 and characterize the optimal contest under the objective function (2). By Theorem 1 and Remark 1, there exists a contest rule and an equilibrium for this contest rule that leads to an arbitrary expected winning probability profile. However, a closed-form solution to the optimal contest rule cannot be obtained in general when the optimal contest involves three or more contestants who place a positive bid with positive probability.[10] As a result, we focus on the expected winning probability profile $\boldsymbol{p}^e \equiv (p_1^e, \ldots, p_n^e)$ and expected effort profile $\boldsymbol{x}^e \equiv (x_1^e, \ldots, x_n^e)$ when characterizing the optimum.

Example 1 *Suppose that the contest designer aims to maximize the objective function as given by* (2)*. In the optimal contest, contestants' equilibrium winning probabilities are given by*

$$p_i^e = \begin{cases} \frac{1+\lambda}{2\gamma}\left\{v_i - \frac{1}{\tau} \times \left[\left(\sum_{j=1}^{\tau} v_j\right) - \frac{2\gamma}{1+\lambda}\right]\right\} & \text{for } i \leq \tau, \\ 0 & \text{for } i > \tau, \end{cases}$$

where τ indicates the number of contestants who submit a positive bid with positive probability and is given by

$$\tau = \begin{cases} 1 & \text{if } \frac{\gamma}{1+\lambda} \leq \frac{1}{2}(v_1 - v_2), \\ \max\left\{m = 1, \ldots, n \,\middle|\, \sum_{j=1}^{m}(v_j - v_m) < \frac{2\gamma}{1+\lambda}\right\} & \text{if } \frac{\gamma}{1+\lambda} > \frac{1}{2}(v_1 - v_2). \end{cases}$$

The expected equilibrium effort profile in the optimal contest is $\boldsymbol{x}^e = (p_1^e v_1, \ldots, p_n^e v_n)$.

[9] It is obvious that there exists another equilibrium in which all contestants choose to remain inactive.

[10] Despite the lack of a closed-form solution, an algorithm that numerically searches for the optimal contest rule can be developed from the proof of Theorem 1.

3.2 Discussions

In what follows, we elaborate on the implications of Theorem 1. We first discuss the respective roles played by multiplicative biases and additive headstarts in contest design. We then take a closer look at Theorem 1 and further elaborate on its implications.

3.2.1 Multiplicative Biases vs. Additive Headstarts

The literature typically focuses on contest design with a single instrument, either multiplicative biases or headstarts. Kirkegaard (2012); Franke, Leininger, and Wasser (2018); and Zhu (2021) show that in a revenue-maximizing all-pay auction, it is generally optimal to employ both. The following can directly be inferred from Theorem 1 and its proof.

Remark 2 *The two steps in the sketch proof of Theorem 1 imply that in general, the optimum requires a combination of multiplicative biases (Step I) and additive headstarts (Step II) for a general contest objective described by Assumption 1.*

A proper combination of the two instruments allows the contest to achieve the frontier of feasible expected effort profile. However, the same does not hold in generalized lottery contests with ratio-form contest success functions. Consider a contest in which one's winning probability is given by

$$p_i(x_i, \boldsymbol{x}_{-i}) = \begin{cases} \dfrac{\alpha_i f(x_i) + \beta_i}{\sum_{j=1}^{n}[\alpha_j f(x_j) + \beta_j]} & \text{if } \sum_{j=1}^{n}[\alpha_j f(x_j) + \beta_j] > 0, \\ \dfrac{1}{n} & \text{if } \sum_{j=1}^{n}[\alpha_j f(x_j) + \beta_j] = 0, \end{cases} \tag{3}$$

where $f(\cdot)$ is twice differentiable, with $f(0) = 0$, $f'(x_i) > 0$, and $f''(x_i) \leq 0$ for all $x_i > 0$. Fu and Wu (2020) establish the following result.

Remark 3 ***(Fu and Wu, 2020, Theorem 2)*** *Suppose that the CSF is given as in* (3) *and that Assumption 1 is satisfied. Then the optimum can always be achieved by choosing only multiplicative biases* $\boldsymbol{\alpha}$ *and setting headstarts* $\boldsymbol{\beta}$ *to zero.*[11]

The contrast between Remarks 2 and 3 demonstrates that headstarts play *different* roles in all-pay auctions and generalized lottery contests. By Remark 3, headstarts are not required to optimize generalized lottery contests. As shown by Fu and Wu (2020), for any contest rule that involves positive headstarts, one can always construct an alternative rule with zero

[11] It should be noted that we do not allow for negative headstarts. Drugov and Ryvkin (2017) allow for negative headstarts and show that a deviation from zero headstarts can locally improve the performance of the contest, depending on the sign of the third derivative of the effort cost function.

headstarts that induces the same equilibrium winning probability profile and strictly higher effort. However, an all-pay auction would invoke headstarts in the optimum. The sketch proof of Theorem 1 reveals the logic: In the first step, we resort to multiplicative biases $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ to induce a given equilibrium winning probability profile. We can then further incentivize the contestant with the highest winning probability by giving additive headstarts to his opponents, as in the second step. This occurs because of the perfectly discriminatory nature of all-pay auctions: The headstarts awarded to underdogs simply force the favorite to shift up the distribution of his effort, which perfectly offsets the headstarts and preserves all contestants' winning odds. This is impossible in a noisy contest that leads to a pure-strategy equilibrium, given the probabilistic nature of the winner-selection mechanism (3).

3.2.2 Full Surplus Extraction in All-pay Auctions

Franke, Leininger, and Wasser (2018) show that a proper combination of multiplicative biases and additive headstarts can achieve first-best result when the designer aims to maximize expected total effort.[12] Theorem 1, together with Remark 1, implies that their result extends to a large class of objective functions as described by Assumption 1. To see this, note that a contestant $i \in \mathcal{N}$ can always guarantee himself a payoff of at least zero by investing zero effort. As a result, in every equilibrium of every contest (i.e., with an arbitrary CSF), the expected payoff of contestant i must be nonnegative, i.e., $x_i^e \leq p_i^e v_i$. By Theorem 1 and Remark 1, with an appropriately designed contest rule and tie-breaking rule, every prize allocation that induces $x_i^e = p_i^e v_i$ for each contestant $i \in \mathcal{N}$ can be implemented. This implies immediately that all-pay auctions dominate *any* other contest mechanism—e.g., the generalized lottery contest specified in (3)—in terms of the resultant (expected) effort x_i^e.

Remark 4 *Suppose that Assumption 1 is satisfied. For any other form of contest that induces a pure-strategy equilibrium (e.g., a generalized lottery contest), there exists an all-pay auction with a CSF as specified in* (1) *that generates a higher payoff for the contest designer.*

A handful of studies examine the comparison between all-pay auctions and Tullock contests—e.g., Fang (2002); Epstein, Mealem, and Nitzan (2011); Franke, Kanzow, Leininger, and Schwartz (2014); and Franke, Leininger, and Wasser (2018). Our analysis sheds light on

[12] The first-best expected total effort is obviously $\max\{v_1, \ldots, v_n\}$. Attaining first best requires that the strongest player win the contest with certainty. Similar to our Remark 1, Franke, Leininger, and Wasser (2018) show in their Proposition 4.7 that first best can be achieved if the designer is able to manipulate the tie-breaking in favor of the strongest player. In addition, Franke, Leininger, and Wasser (2018) show in their Proposition 4.8 that, with a symmetric tie-breaking rule, the contest designer can generate an amount of expected total effort that is arbitrarily close to the first best.

this literature: It accommodates a broader design objective and establishes the dominance of all-pay auctions over a larger class of contest mechanisms, i.e., *any* contest that induces pure-strategy bidding.

4 Concluding Remarks

In this paper, we consider the optimal design of complete-information all-pay auctions with general contest objectives. We apply the indirect approach suggested by Fu and Wu (2020) and Deng, Fu, and Wu (2021) and characterize the general properties of the optimal contest. In particular, we show that both instruments will be used in the optimum in general. Further, an optimally designed all-pay auction can achieve full surplus extraction for a large class of objectives.

Our framework leaves room for future extensions. We focus on expected efforts as a measure of contestants' incentives. Maximizing the expected winner's effort is common in the auction literature (e.g., Moldovanu and Sela, 2006) and has recently gained increasing attention in studies of contests (e.g., Baye and Hoppe, 2003; Serena, 2017; Fu and Wu, 2020, 2021; Wasser and Zhang, 2021). Because contestants typically employ a mixed strategy in a complete-information all-pay auction, incorporating the expected winner's effort into the design objective causes substantial nuances and is analytically challenging within our framework: The expected winner's effort is defined over the entire joint distribution of efforts based on the whole profile of contestants' mixed strategies. In contrast, each contestant's expected effort depends on his own bidding strategy.[13] We leave exploration of this possibility to future research.

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[13] This nuance does not arise in a generalized lottery contest—e.g., Fu and Wu (2020)—because contestants play pure strategies.

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Appendix: Proofs

Proof of Theorem 1

Proof. Denote the expected equilibrium winning probability profile we would like to induce by $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*) \in \Delta^{n-1}$. Without loss of generality, let us assume $p_1^* \geq \cdots \geq p_n^*$. We apply the equilibrium characterization in Theorem 2 in Baye, Kovenock, and De Vries (1996) to prove the result for the case $p_1^* > p_2^* \geq \cdots \geq p_n^*$. The case $p_1^* = p_2^* \geq \cdots \geq p_n^*$ can be proved in a similar way by invoking Theorem 1 in Baye, Kovenock, and De Vries (1996).

Step I (Introducing Multiplicative Biases): We show that, fixing an arbitrary $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*) \in \Delta^{n-1}$ such that $p_i^* \neq 1$ for all $i \in \mathcal{N}$, we can construct a set of multiplicative biases $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$ to induce $\boldsymbol{p}^*$. To proceed, we set $\boldsymbol{\beta} = \mathbf{0}$ and choose $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ such that $\hat{v}_1 > \hat{v}_2 = \cdots = \hat{v}_n > 0$, where $\hat{v}_i := \alpha_i v_i$ for all $i \in \mathcal{N}$. The prize valuation $\hat{v}_2$ can be an arbitrary positive real number and $\hat{v}_1$—or equivalently, the ratio $\hat{v}_2/\hat{v}_1$—will be determined later in the proof.

Let $\widehat{G}_i(x_i)$ denote the CDF representing the equilibrium mixed-strategy of player i. By Theorem 2 in Baye, Kovenock, and De Vries (1996), there exists a continuum of equilibria of the unbiased all-pay auction with valuations $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{\alpha}} \equiv (1, \ldots, 1)$, which is fully characterized by a set of cutoffs (free parameters) $\boldsymbol{b} \equiv (b_1, \ldots, b_n)$ that satisfy $0 = b_1 = b_2 \leq \cdots \leq b_n \leq \hat{v}_2$. In equilibrium, player i stays inactive with some probability and bids continuously over $(b_i, \hat{v}_2]$ with complementary probability. For notational convenience, let $c_i := \frac{\hat{v}_1 - \hat{v}_2 + b_i}{\hat{v}_1}$ for all $i \in \mathcal{N}$. Baye, Kovenock, and De Vries (1996) show that the following CDFs $\langle \widehat{G}_1(x_1), \ldots, \widehat{G}_n(x_n) \rangle$ constitute a mixed-strategy equilibrium of the unbiased all-pay auction:

$$
\begin{aligned}
\forall x \in \big[b_n, \hat{v}_2\big]: \quad & \widehat{G}_1(x) = \frac{x}{\hat{v}_2} \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{2-n}{n-1}}; \\
& \widehat{G}_i(x) = \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{1}{n-1}}, \ i \in \big\{2, 3, \ldots, n\big\}; \\
\forall x \in \big[b_j, b_{j+1}\big), j \in \{3, \ldots, n-1\}: \quad & \widehat{G}_1(x) = \frac{x}{\hat{v}_2} \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{2-j}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}; \\
& \widehat{G}_i(x) = \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{1}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}, \ i \in \big\{2, \ldots, j\big\}; \\
& \widehat{G}_k(x) = c_k^{\frac{1}{k-1}} \prod_{s>k} c_s^{-\frac{1}{(s-1)(s-2)}}, \ k \in \big\{j+1, \ldots, n\big\}; \\
\forall x \in \big[0, b_3\big]: \quad & \widehat{G}_1(x) = \frac{x}{\hat{v}_2} \prod_{k>2} c_k^{-\frac{1}{(k-1)(k-2)}};
\end{aligned}
$$

$$\widehat{G}_2(x) = \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right] \prod_{k>2} c_k^{-\frac{1}{(k-1)(k-2)}};$$

$$\widehat{G}_k(x) = c_k^{\frac{1}{k-1}} \prod_{s>k} c_s^{-\frac{1}{(s-1)(s-2)}},\ k \in \{3, \ldots, n\}.$$

According to the above equilibrium characterization, we can calculate contestant i's expected effort, which we denote by $\hat{x}_i^e$. For notational convenience, define $\mu := \hat{v}_2/\hat{v}_1 < 1$ and let $b_{n+1} := \hat{v}_2$. The expected effort of player 1 can then be derived as

$$\begin{aligned}
\hat{x}_1^e &= \int_0^{\hat{v}_2} x d\widehat{G}_1(x) \\
&= \hat{v}_2 - \sum_{j=2}^{n} \left[\int_{b_j}^{b_{j+1}} \widehat{G}_1(x) dx\right] \\
&= \hat{v}_2 - \sum_{j=2}^{n} \left[\int_{b_j}^{b_{j+1}} \frac{x}{\hat{v}_2} \left(\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right)^{\frac{2-j}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dx\right] \\
&= \hat{v}_2 - \sum_{j=2}^{n} \left[\int_{c_j}^{c_{j+1}} \frac{\hat{v}_1^2}{\hat{v}_2} (y - 1 + \mu) y^{\frac{2-j}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dy\right] \\
&= \hat{v}_2 - \frac{\hat{v}_1^2}{\hat{v}_2} \sum_{j=2}^{n} \left\{\left[\frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right) - (1-\mu)(j-1)\left(c_{j+1}^{\frac{1}{j-1}} - c_j^{\frac{1}{j-1}}\right)\right] \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right\}.
\end{aligned} \tag{4}$$

Similarly, for contestant $i \in \{2, \ldots, n\}$, we have that

$$\begin{aligned}
\hat{x}_i^e &= \int_{b_i}^{\hat{v}_2} x d\widehat{G}_i(x) \\
&= \hat{v}_2 - b_i \widehat{G}_i(b_i) - \sum_{j=i}^{n} \left[\int_{b_j}^{b_{j+1}} \widehat{G}_i(x) dx\right] \\
&= \hat{v}_2 - b_i \widehat{G}_i(b_i) - \sum_{j=i}^{n} \left[\int_{b_j}^{b_{j+1}} \left(\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right)^{\frac{1}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dx\right] \\
&= \hat{v}_2 - b_i \widehat{G}_i(b_i) - \sum_{j=i}^{n} \left[\hat{v}_1 \int_{c_j}^{c_{j+1}} y^{\frac{1}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dy\right] \\
&= \hat{v}_2 - b_i \widehat{G}_i(b_i) - \sum_{j=i}^{n} \left[\hat{v}_1 \frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right) \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right]
\end{aligned}$$

$$= \hat{v}_2 - \hat{v}_1(c_i - 1 + \mu)c_i^{\frac{1}{i-1}} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} - \sum_{j=i}^{n} \left[\hat{v}_1 \frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right]. \tag{5}$$

By Theorem 2 in Baye, Kovenock, and De Vries (1996), player 1 earns an expected payoff of $\hat{v}_1 - \hat{v}_2$, while every other player receives an expected payoff of zero in the transformed unbiased all-pay auction with valuations $\hat{\boldsymbol{v}} \equiv (\hat{v}_1, \dots, \hat{v}_n)$, i.e.,

$$\hat{p}_1^e \hat{v}_1 - \hat{x}_1^e = \hat{v}_1 - \hat{v}_2, \tag{6}$$

$$\hat{p}_i^e \hat{v}_i = \hat{x}_i^e, i \in \{2, \dots, n\}, \tag{7}$$

where $\hat{p}_i^e$ is contestant i's expected winning probability.

Combining (4) and (6), we can obtain $\hat{p}_1^e$ as a function of μ and $\boldsymbol{c} \equiv (c_1, \dots, c_n)$:

$$\begin{aligned}\hat{p}_1^e(\mu, \boldsymbol{c}) &= \frac{\hat{x}_1^e}{\hat{v}_1} + \frac{\hat{v}_1 - \hat{v}_2}{\hat{v}_1} \\ &= 1 - \frac{1}{\mu} \sum_{j=2}^{n} \left\{ \left[\frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) - (1-\mu)(j-1) \left(c_{j+1}^{\frac{1}{j-1}} - c_j^{\frac{1}{j-1}} \right) \right] \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right\}.\end{aligned} \tag{8}$$

Similarly, combining (5) and (7), for $i \in \{2, \dots, n\}$, we have that

$$\begin{aligned}\hat{p}_i^e(\mu, \boldsymbol{c}) &= \frac{\hat{x}_i^e}{\hat{v}_2} \\ &= 1 - \frac{1}{\mu} (c_i - 1 + \mu) c_i^{\frac{1}{i-1}} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} - \sum_{j=i}^{n} \left[\frac{1}{\mu} \frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right].\end{aligned} \tag{9}$$

To prove the statement we made at the beginning of Step I, it suffices to construct $\mu \in (0,1)$ and $\boldsymbol{c} \equiv (c_1, \dots, c_n)$, with $1 - \mu = c_1 = c_2 \leq \dots \leq c_n \leq 1$, such that $p_i^* = \hat{p}_i^e(\mu, \boldsymbol{c})$ for all $i \in \mathcal{N}$. To proceed, it is useful to prove an intermediate result.

Lemma 2 *For any $i \geq 3$, $\hat{p}_i^e(\mu, \boldsymbol{c})$ is strictly decreasing in c_i.*

Proof. $\hat{p}_i^e(\mu, \boldsymbol{c})$ in (9) can be rewritten as

$$\hat{p}_i^e(\mu, \boldsymbol{c}) = \left\{ 1 - \sum_{j=i+1}^{n} \left[\frac{1}{\mu} \frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right] - \frac{1}{\mu} \frac{i-1}{i} c_{i+1}^{\frac{i}{i-1}} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} \right\}$$

$$+\frac{1}{\mu}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}}\left[\frac{i-1}{i}c_i^{\frac{i}{i-1}}-(c_i-1+\mu)c_i^{\frac{1}{i-1}}\right].$$

Therefore, it suffices to show that

$$h(c_i):=\frac{i-1}{i}c_i^{\frac{i}{i-1}}-(c_i-1+\mu)c_i^{\frac{1}{i-1}}$$

is decreasing in c_i. Simple algebra yields that

$$h'(c_i)=-\frac{1}{i-1}\left[c_i-(1-\mu)\right]c_i^{\frac{2-i}{i-1}}\leq 0,$$

where the inequality follows from $c_i \geq c_2 \equiv 1-\mu$ for $i \geq 3$. This concludes the proof. ■

We are now ready to prove the statement we made at the beginning of Step I. Note that $\hat{p}_i^e(\mu, \boldsymbol{c})$ is a function of μ and $(c_i, \dots, c_n)$, and is independent of $(c_1, \dots, c_{i-1})$ for $i \geq 3$. With slight abuse of notation, we write $\hat{p}_i^e(\mu, \boldsymbol{c})$ in (9) as $\hat{p}_i^e(\mu, c_i, \dots, c_n)$ in what follows.

Fix $\boldsymbol{p}^* \equiv (p_1^*, \dots, p_n^*)$. We recursively define a set of functions $\left\{\tilde{c}_i(\mu)\right\}_{i=1}^n$ and a function $\psi(\mu)$ as follows:

Step 0: Set $\psi(\mu)=1$, and define $\tilde{c}_n(\mu)$ as

$$\tilde{c}_n(\mu):=\begin{cases}1-\mu & \text{if } \hat{p}_n^e(\mu,1-\mu)<p_n^*,\\ \text{The unique solution to } \hat{p}_n^e(\mu,c_n)=p_n^* & \text{otherwise.}\end{cases} \tag{10}$$

Lemma 2, together with the fact that $\hat{p}_n^e(\mu,1)=0$, implies that $\tilde{c}_n(\mu)$ is well defined and $\tilde{c}_n(\mu)\in[1-\mu,1]$. If $\hat{p}_n^e(\mu,1-\mu)<p_n^*$, define $\tilde{c}_i(\mu)=1-\mu$ for $i\geq 3$, update $\psi(\mu)=n$, and move to Step $n-2$. Otherwise, we proceed to Step 1.

Step $j\in\{1,\dots,n-3\}$: Define $\tilde{c}_{n-j}(\mu)$ as

$$\tilde{c}_{n-j}(\mu):=\begin{cases}1-\mu, \text{ if } \hat{p}_{n-j}^e\left(\mu,1-\mu,\tilde{c}_{n-j+1}(\mu),\dots,\tilde{c}_n(\mu)\right)<p_{n-j}^*,\\ \text{The unique solution to } \hat{p}_{n-j}^e\left(\mu,c_{n-j},\tilde{c}_{n-j+1}(\mu),\dots,\tilde{c}_n(\mu)\right)=p_{n-j}^*, \text{ otherwise.}\end{cases}$$

Lemma 2, together with the fact that $\hat{p}_{n-j}^e\big(\mu,\tilde{c}_{n-j+1}(\mu),\tilde{c}_{n-j+1}(\mu),\dots,\tilde{c}_n(\mu)\big)=p_{n-j+1}^*\leq p_{n-j}^*$, implies that $\tilde{c}_{n-j}(\mu)$ is well defined and $\tilde{c}_{n-j}(\mu)\in\left[1-\mu,\tilde{c}_{n-j+1}(\mu)\right]$. If $\hat{p}_{n-j}^e\big(\mu,1-\mu,\tilde{c}_{n-j+1}(\mu),\dots,\tilde{c}_n(\mu)\big)<p_{n-j}^*$, define $\tilde{c}_i(\mu)=1-\mu$ for $i\in\{3,\dots,n-j\}$, update $\psi(\mu)=n-j$, and move to Step $n-2$. Otherwise, we proceed to Step $j+1$.

Step $n-2$: Set $\tilde{c}_1(\mu)=\tilde{c}_2(\mu)=1-\mu$.

Let $\tilde{\boldsymbol{c}}(\mu) := \big(\tilde{c}_1(\mu), \ldots, \tilde{c}_n(\mu)\big)$. Fixing μ, we can calculate $\tilde{\boldsymbol{c}}(\mu)$ and $\psi(\mu)$ through the steps above. To complete the proof, it suffices to show that there exists $\mu \in (0,1]$ such that $\hat{p}_1^e\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = p_1^*$ and $\psi(\mu) = 1$.

We first show that there exists a solution to $\hat{p}_1^e\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = p_1^*$. It can be verified that $\tilde{\boldsymbol{c}}(\mu)$ is continuous on the interval $\mu \in (0,1]$. Moreover, it follows from Equation (10) and the construction in Step 0 that $\tilde{\boldsymbol{c}}(\mu) = (1-\mu, \ldots, 1-\mu)$ when μ is sufficiently small; together with Equation (8), we have that $\lim_{\mu \searrow 0} \hat{p}_1^e\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = 1 > p_1^*$. Therefore, it suffices to show that $\hat{p}_1^e\big(1, \tilde{\boldsymbol{c}}(1)\big) < p_1^*$. We consider two cases:

Case (a): $\psi(1) = 1$. Then $\tilde{c}_2 = 1 - \mu = 0$, and thus $\hat{p}_1^e\big(1, \tilde{\boldsymbol{c}}(1)\big) = \hat{p}_2^e\big(1, \tilde{c}_2(1), \ldots, \tilde{c}_n(1)\big)$ by (8) and (9). Moreover, we have that $\hat{p}_j^e\big(1, \tilde{c}_j(1), \ldots, \tilde{c}_n(1)\big) = p_j^*$ for all $j \geq 3$. Therefore, we have that

$$\hat{p}_1^e\big(1, \tilde{\boldsymbol{c}}(1)\big) = \hat{p}_2^e\big(1, \tilde{c}_2(1), \ldots, \tilde{c}_n(1)\big) = \frac{p_1^* + p_2^*}{2} < p_1^*.$$

Case (b): $\psi(1) \neq 1$. For notational convenience, let $\kappa := \psi(1) \geq 3$. By (8), (9), and the definition of $\psi(\cdot)$, $\hat{p}_j^e\big(1, \tilde{c}_j(1), \ldots, \tilde{c}_n(1)\big) = p_j^*$ for all $j \geq \kappa + 1$ and $\hat{p}_1^e\big(1, \tilde{\boldsymbol{c}}(1)\big) = \cdots = \hat{p}_\kappa^e\big(1, \tilde{c}_\kappa(1), \ldots, \tilde{c}_n(1)\big)$. By the same argument used in Case (a), we have that

$$\hat{p}_1^e\big(1, \tilde{\boldsymbol{c}}(1)\big) = \frac{\sum_{i=1}^{\kappa} p_i^*}{\kappa} < p_1^*.$$

Denote the solution to $\hat{p}_1^e\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = p_1^*$ by μ^*. It remains to show that $\kappa^* := \psi(\mu^*) = 1$. Suppose to the contrary that $\kappa^* \geq 3$. Then

$$\hat{p}_j^e\big(\mu^*, \tilde{c}_j(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) = p_j^* \text{ for all } j \geq \kappa^* + 1,$$

and

$$\hat{p}_2^e\big(\mu^*, \tilde{c}_2(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) = \cdots = \hat{p}_{\kappa^*}^e\big(\mu^*, \tilde{c}_{\kappa^*}(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) < p_{\kappa^*}^*,$$

by (9) and the definition of $\psi(\cdot)$. Therefore, we have that

$$\begin{aligned}
\hat{p}_1^e\big(\mu^*, \tilde{\boldsymbol{c}}(\mu^*)\big) &= 1 - \sum_{i=2}^{\kappa^*} \hat{p}_i^e\big(\mu^*, \tilde{c}_i(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) - \sum_{i=\kappa^*+1}^{n} \hat{p}_i^e\big(\mu^*, \tilde{c}_i(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) \\
&> 1 - (\kappa^* - 1) p_{\kappa^*}^* - \sum_{i=\kappa^*+1}^{n} p_i^* \\
&\geq 1 - \sum_{i=2}^{\kappa^*} p_i^* - \sum_{i=\kappa^*+1}^{n} p_i^* = p_1^*,
\end{aligned}$$

which contradicts $\hat{p}_1^e\big(\mu^*, \tilde{\boldsymbol{c}}(\mu^*)\big) = p_1^*$. Therefore, $\hat{p}_i^e\big(\mu^*, \tilde{\boldsymbol{c}}(\mu^*)\big) = p_i^*$ for all $i \in \mathcal{N}$ and $\psi(\mu^*) = 1$.

Step II (Introducing Additive Headstarts): Denote the set of multiplicative biases we constructed in Step I that leads to $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ by $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$. Let $\hat{v}_i^* := \alpha_i^* v_i$ for all $i \in \mathcal{N}$ and denote the corresponding equilibrium strategy profile under $\langle \hat{\boldsymbol{v}}^*, \hat{\boldsymbol{\alpha}} \rangle := \langle (\hat{v}_1^*, \ldots, \hat{v}_n^*), (1, \ldots, 1) \rangle$ and zero headstarts by $\langle \widehat{G}_1^*(x_1), \ldots, \widehat{G}_n^*(x_n) \rangle$. Denote player i's expected equilibrium effort by $\hat{x}_i^{e*}$.

By Lemma 1, there exists an equilibrium strategy profile under $\langle \boldsymbol{v}, \boldsymbol{\alpha}^* \rangle$ and zero headstarts, which we denote by $\langle G_1^*(x_1), \ldots, G_n^*(x_n) \rangle$, that leads to the profile of the expected winning probabilities $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$; moreover, contestant i's expected effort in the equilibrium, which we denote by x_1^{e*}, satisfies

$$
\begin{aligned}
x_1^{e*} &= \frac{\hat{x}_1^{e*}}{\alpha_1^*} = p_1^* \frac{\hat{v}_1^*}{\alpha_1^*} - \frac{\hat{v}_1^* - \hat{v}_2^*}{\alpha_1^*} = p_1^* v_1 - \frac{\alpha_1^* v_1 - \alpha_2^* v_2}{\alpha_1^*} < p_1^* v_1, \\
x_i^{e*} &= \frac{\hat{x}_i^{e*}}{\alpha_i^*} = p_i^* \frac{\hat{v}_i^*}{\alpha_i^*} = p_i^* v_i, \text{ for } i \in \{2, \ldots, n\}.
\end{aligned}
$$

In fact, $\langle G_1^*(x_1), \ldots, G_n^*(x_n) \rangle = \langle \widehat{G}_1^*(\alpha_1^* x_1), \ldots, \widehat{G}_n^*(\alpha_n^* x_n) \rangle$.

Next, we introduce additive headstarts to the contest rule. To be more specific, consider the following contest rule $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger)$:

$$
(\alpha_i^\dagger, \beta_i^\dagger) := \begin{cases} (\alpha_1^*, 0) & \text{for } i = 1, \\ (\alpha_i^*, \alpha_1^* v_1 - \alpha_2^* v_2) & \text{for } i \in \{2, \ldots, n\}. \end{cases}
$$

It can be verified that a mixed-strategy equilibrium exists in the all-pay auction under the contest rule $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger)$, in which player 1 randomizes according to CDF $\widehat{G}_1^*\left(\alpha_1^* x_1 - (\alpha_1^* v_1 - \alpha_2^* v_2)\right)$ and player $i \in \{2, \ldots, n\}$ randomizes according to CDF $\widehat{G}_i^*\left(\alpha_i^* x_i\right)$. It is straightforward to verify that this equilibrium strategy profile again leads to the expected equilibrium winning probability profile $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ and contestant i's expected effort is $p_i^* v_i$, which in turn implies that each contestant earns an expected payoff of zero. This concludes the proof. ■