A Remark on the solution to the Lucas-Uzawa model with increasing returns to scale

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Abstract. Concerning the famous Lucas-Uzawa model with increasing returns to scale, [1] gives a closed form solution for the case, where the discount rate is smaller than a specific threshold (modified intrinsic growth rate of human capital in some sense). In this remark, we use the Bellman equation to solve this problem for the case, where the discount rate is bigger than that threshold, and give the closed form solution, which is a corner solution.

Keywords. Lucas-Uzawa model, corner solution, Bellman equation.

In [1], the author considers the problem of Lucas-Uzawa model from the social planner's perspective:

$$
\begin{aligned}
\max \quad & \int_0^\infty e^{-\rho t} C^\beta dt, \\
\text{s.t.} \quad & \dot{K} = aK^\alpha (uH)^\beta H^\gamma - C - \delta K, \\
& \dot{H} = bH(1-u), \\
& u \in [0,1], \\
& K \geq 0, \quad C \geq 0, \\
& K(0) = K_0, \quad H(0) = H_0,
\end{aligned}
$$

where $\alpha \in (0,1)$, $\beta = 1-\alpha$, $\rho > 0, \gamma > 0, a > 0, b > 0, \delta \geq 0, K_0 > 0, S_0 > 0$ are all constants. Here and in what follows, in general, for simplicity , for any dynamic variable x, we denote x, instead of $x(t)$.

By letting

$$S = H^{(\beta+\gamma)/\beta}, \quad R = uS, \quad r = \frac{b(\beta+\gamma)}{\beta},$$

this problem can be transformed equivalently to problem $\mathbb{P}^*$:

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} C^\beta dt, \\ \text{s.t.} \quad & \dot{K} = aK^\alpha R^\beta - C - \delta K, \\ & \dot{S} = r(S - R), \\ & 0 \le R \le S, \\ & K \ge 0, \quad C \ge 0, \\ & K(0) = K_0, \quad S(0) = S_0, \end{aligned}$$

where $\alpha \in (0,1)$, $\beta = 1 - \alpha$, $\rho > 0, a > 0, r > 0, \delta \ge 0, K_0 > 0, S_0 > 0$ are all constants.

The author's result is as follows: if $\rho < \beta r$, then, the problem has no solution in that there exists a feasible path enabling the objective functional to be ∞; if $\beta r < \rho < r$, then, there is an interior solution, and the author gives an analytical form of the solution.

In what follows, we prove that if $\rho = \beta r$, then, there is also no solution; if $\rho \ge r$, then, there is a corner solution.

- **The case $\rho = \beta r$**

In this case, the problem has no solution. In fact, take $\varepsilon \in (0, r)$, and let

$$\begin{aligned} & R = R_0 e^{\varepsilon t}, \quad S = S_0 e^{\varepsilon t}, \\ & C = K = \left(A e^{-\beta(\delta+1)t} + B e^{\beta \varepsilon t} \right)^{1/\beta}, \end{aligned}$$

where

$$R_0 = S_0 \frac{r - \varepsilon}{r}, \quad A = K_0^\beta - B, \quad B = \frac{R_0^\beta}{\varepsilon + \delta + 1}.$$

The value of the objective functional, corresponding to this path, is

$$\begin{aligned} V(\varepsilon) &= \int_0^\infty e^{-\rho t} \left[A e^{-\beta(\delta+1)t} + B e^{\beta \varepsilon t} \right] dt \\ &= \frac{A}{\rho + \beta(\delta + 1)} + B \int_0^\infty e^{(\beta\varepsilon - \rho)t} dt \\ &= \frac{A}{\rho + \beta(\delta + 1)} + \frac{B}{\rho - \beta\varepsilon}, \end{aligned}$$

letting $\varepsilon \to r$, we get $V(\varepsilon) \to \infty$.

- **The case $\rho \ge r$**

*It can also be seen as a two-sector growth problem with natural resources owning linear regeneration function.

In this case, the unique solution is the corner solution $(*)$:

$$
\begin{aligned}
C &= \frac{\rho+\beta\delta}{\alpha}\left[S_0^\beta \frac{a}{\tau}\left(1-e^{-\tau\beta t}\right)+K_0^\beta e^{-\tau\beta t}\right]^{1/\beta}, \\
K &= \left[S_0^\beta \frac{a}{\tau}\left(1-e^{-\tau\beta t}\right)+K_0^\beta e^{-\tau\beta t}\right]^{1/\beta}, \\
R &\equiv S_0, \\
S &\equiv S_0,
\end{aligned}
$$

where

$$\tau = \frac{\rho+\delta}{\alpha}.$$

Proof. The Bellman equation for $\mathbb{P}$ is

$$\rho V(K,S) = \max_{0\le C, 0\le R\le S}\left\{C^\beta + V_K\left(aK^\alpha R^\beta - \delta K - C\right) + V_S r(S-R)\right\},$$

which can be easily verified to have unique solution

$$V(K,S) = \sigma^{-\alpha}\left(K^\beta + \frac{a\beta}{\rho}S^\beta\right),$$

where

$$\sigma = \frac{\rho+\beta\delta}{\alpha},$$

and the corresponding optimal Markovian strategy is

$$C = \sigma K, \quad R = S,$$

and the optimal path induced by this strategy is just $(*)$. The proof is completed.

References

[1] Ryoji Hiraguchi. A solution to the Lucas-Uzawa model with increasing returns to scale: Note. Economic Modelling 26 (2009) 831-834.